CP1: Mechanics and Special Relativity

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Contents

1. *Mechanics I*

This chapter aims to cover the basics of mechanics, including:

- Newton's Laws and Energy Conservation
- Collisions
- Resisted Motion
- Variable Mass problems

Most of the material covered here will not be unfamiliar to students who have studied physics in high-school, though it may be expressed in a slightly different way, such as using vector notation in collisions. The mathematics required is quite simple, only requiring basic knowledge of integration and linear algebra.

1.1 Newton's Laws and Energy Conservation

Most of the physics dealt with in this chapter relies on these two concepts, and their consequences. It is essential that any physics student is very much at ease with these.

1.1.1 Newton's Laws

All physics students should already be very familiar with Newton's Laws of Motion. They are as follows:

1. Every body moves in a state of uniform motion unless acted upon by a net force
2. The rate of change of momentum is equal to the applied force:

$$\sum \underline{F} = \frac{d\underline{p}}{dt} = \underline{v}\frac{dm}{dt} + m\frac{d\underline{v}}{dt} \tag{1.1}$$

3. When a body exerts a force on a second body, the second body simultaneously exerts a force equal and opposite to the second body.

Most students may not have seen (1.1) in the form shown above; generally, this will reduce to $\underline{F} = m\frac{d\underline{v}}{dt}$ for problems that do not involved variable mass.. However, (1.1) does lead to an interesting result: *in an isolated system, total momentum is conserved.*

$$\sum \underline{F} = 0$$
$$\rightarrow \frac{d\underline{p}}{dt} = 0$$

This means that p_r (the linear momentum) is a conserved quantity.

In general, we only use Newton's Laws for simple systems, as it is much more effective to use other methods when the system becomes more complicated.

1.1.2 Energy Conservation

Most forces are what we term *non-conservative* forces. Energy is lost from the system as a result of the force, usually dissipated as heat or light energy. However, there is a special class of forces called (you guessed it!) *conservative forces.* In this case, the work done to move an object between two points in space is *independent of the path taken.* An everyday example of this is gravity. You may take any path from point a to b, but the work done will always be given by:

$$U_{a\rightarrow b} = U(b) - U(a) \tag{1.2}$$

As a result, for a conservative field, we can write the force as a gradient of some scalar potential:

$$\underline{F} = -\nabla U \tag{1.3}$$

This result is proven in the notes on CP4 in the section on conservative forces.

The law of the conservation of energy states that *in any closed system, subject to no external non-conservative forces, energy is conserved.* This can be used as a powerful analytic tool, even when the system is subject to external forces. For example, one can find the about of energy dissipated by a resistive force by equating the energy before and after; we know that the difference must be equal to the work done by the non-conservative force.

1.2 Collisions

These should be the types of problems that most people will already be familiar with at this stage, but we would recommend reading this section in any case as there may be some elements not previously encountered. There are a couple of crucial points to remember for collisions:

- In all collisions subject to no external forces, momentum is conserved
- In elastic collisions, energy is conserved
- In in-elastic collisions, we have:

$$\text{Energy After} = \epsilon \cdot (\text{Energy Before}) \tag{1.4}$$

$$\epsilon = \left| \frac{v_2 - v_1}{u_1 - u_2} \right| \tag{1.5}$$

  The quantity ϵ is known as the *coefficient of restitution*. It is a measure of the amount of energy that is conserved through the collision, and takes values $0 < \epsilon < 1$. For elastic collisions, $\epsilon = 1$, and for totally in-elastic collisions, $\epsilon = 0$.

- For an oblique collision, the motion is only changed along the common normal between two objects

We can use these results to derive a useful result for elastic collisions. Consider two objects m_1 and m_2 that undergo a head-on collision. By the conservation of energy and momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$m_1(v_1 - u_1) = m_2(u_2 - v_2)$$

$$m_1{u_1}^2 + m_2{u_2}^2 = m_1{v_1}^2 + m_2{v_2}^2$$
$$m_1({v_1}^2 - {u_1}^2) = m_2({u_2}^2 - {v_2}^2)$$
$$m_1(v_1 - u_1) \cdot (v_1 + u_1) = m_2(u_2 - v_2) \cdot (u_2 + v_2)$$

Hence, we obtain:

$$v_1 - v_2 = u_2 - u_1 \tag{1.6}$$

This means that in a elastic collision, relative velocity is maintained throughout.

1.2.1 Centre of Mass frame

This frame, referred to in short as CM, is a very useful tool for solving collision problems. In this case, we define all quantities relative to the centre of mass, which is defined as:

$$\underline{r}_{cm} = \frac{\sum_i m_i \underline{r}_i}{\sum_i m_i} \tag{1.7}$$

We then use the Galilean transformations to move into the CM frame.

$$\underline{u}_i' = \underline{u}_i - \dot{\underline{r}}_{cm}$$
$$\underline{u}_i = \underline{u}_i' + \dot{\underline{r}}_{cm}$$

Generally, one notates CM frame quantities using a prime.

The CM frame is useful for three main reasons:

- There is net zero momentum in the CM
- The speeds of the particles before and after the collision are the same.

$$\begin{aligned}
m_1\underline{u}_1' + m_2\underline{u}_2' &= 0 \\
m_1\underline{v}_1' + m_2\underline{v}_2' &= 0 \\
\underline{u}_2' - \underline{u}_1' &= \underline{v}_1' - \underline{v}_2' \\
-\frac{m_1}{m_2}\underline{u}_1' - \underline{u}_1' &= \underline{v}_1' + \frac{m_1}{m_2}\underline{v}_1' \\
-\underline{u}_1'\left(1 + \frac{m_1}{m_2}\right) &= \underline{v}_1'\left(1 + \frac{m_1}{m_2}\right) \\
\rightarrow |\underline{u}_i| &= |\underline{v}_i| \qquad (1.8)
\end{aligned}$$

- The particles leave an oblique collision at the same angle in the centre of mass frame. This is because angles are transformed in the CM frame; we need to use trigonometry to convert back to the lab frame.

Consider the following example. A particle of mass $2m$ travelling at an initial speed u collides inelastically with a stationary particle of mass m. *What is the velocity of the second particle after the collision?*

First, we want to transform to the centre of mass frame:

$$\underline{v}_{cm} = \frac{2u}{3}\begin{pmatrix}1\\0\end{pmatrix}$$

The initial velocities in the CM are thus:

$$\begin{aligned}
\underline{u}_1' &= \frac{u}{3}\begin{pmatrix}1\\0\end{pmatrix} \\
\underline{u}_2' &= \frac{2u}{3}\begin{pmatrix}-1\\0\end{pmatrix}
\end{aligned}$$

Consider Figure (1.1).

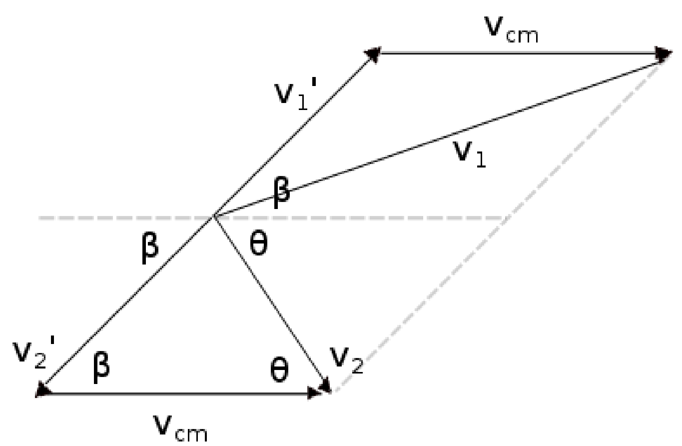

Figure 1.1: Transforming between the CM and lab frames

We can use (1.5) to write that:

$$\underline{v}_1' = \epsilon \cdot \frac{u}{3}\begin{pmatrix}\cos\beta \\ \sin\beta\end{pmatrix}$$
$$\underline{v}_2' = \epsilon \cdot \frac{2u}{3}\begin{pmatrix}-\cos\beta \\ -\sin\beta\end{pmatrix}$$

Transforming back to the lab frame:

$$\begin{aligned}\underline{v}_1 &= \underline{v}_1' + \underline{v}_{cm} \\ &= \frac{u}{3}\begin{pmatrix}\epsilon\cos\beta + 2 \\ \epsilon\sin\beta\end{pmatrix} \\ \underline{v}_2 &= \underline{v}_2' + \underline{v}_{cm} \\ &= \frac{2u}{3}\begin{pmatrix}-\epsilon\cos\beta + 1 \\ -\epsilon\sin\beta\end{pmatrix}\end{aligned}$$

Using the cosine rule with reference to Figure (1.1):

$$\begin{aligned}{\underline{v}_2'}^2 &= {\underline{v}_2}^2 + {\underline{v}_{cm}}^2 - 2\underline{v}_2\underline{v}_{cm}\cos\theta \\ \underline{v}_2 &= \underline{v}_{cm}\left(\cos\theta \pm \sqrt{\left(\frac{\underline{v}_2'}{\underline{v}_{cm}}\right)^2 - \sin^2\theta}\right) \\ \rightarrow {\underline{v}_2'}^2 &= \underline{v}_{cm}\left(\cos\theta \pm \sqrt{\epsilon^2 - \sin^2\theta}\right)\end{aligned}$$

This is the final velocity of the particle of mass m in the lab frame.

1.2.2 Oblique Scattering

Suppose that a particle of mass m undergoes an elastic collision with an identical stationary particle. Let the angle separating the final velocities of the two particles be θ. By the conservation of energy and momentum:

$$\begin{aligned}{\underline{u}_1}^2 &= {\underline{v}_1}^2 + {\underline{v}_2}^2 \\ \underline{u}_1 &= \underline{v}_1 + \underline{v}_2\end{aligned}$$

Taking the dot product of both sides:

$$\begin{aligned}\underline{u}_1 \cdot \underline{u}_1 &= \underline{v}_1 \cdot \underline{v}_1 + 2\underline{v}_1 \cdot \underline{v}_2 + \underline{v}_2 \cdot \underline{v}_2 \\ {\underline{u}_1}^2 &= {\underline{v}_1}^2 + {\underline{v}_2}^2 + 2\underline{v}_1\underline{v}_2 \cdot \cos(\theta)\end{aligned}$$

$$\rightarrow 2\underline{v}_1\underline{v}_2 \cdot \cos(\theta) = 0 \tag{1.9}$$

This leads to the interesting result that either $\underline{v}_1 = 0$ or $\theta = \frac{\pi}{2}$ after the collision. In the second case, $\underline{u}_1 \cdot \underline{v}_1 > 0$, and hence the first particle is scattered through an angle of less than $\frac{\pi}{2}$.

1.2.3 Max Deflection Angle

Consider an elastic collision between a particle of mass m_1 moving with a velocity u_1 and a stationary particle of mass m_2. By the conservation of momentum:

$$\begin{aligned}m_1\underline{u}_1 &= m_1\underline{v}_1 + m_2\underline{v}_2 \\ \underline{v}_2 &= \frac{m_1}{m_2} \cdot (\underline{u}_1 - \underline{v}_2)\end{aligned}$$

By the conservation of energy:

$$\begin{aligned} m_1\underline{u}_1{}^2 &= m_1\underline{v}_1{}^2 + m_2\underline{v}_2{}^2 \\ m_1\underline{u}_1{}^2 - m_1\underline{v}_1{}^2 &= m_2\underline{v}_2{}^2 \\ m_1\left(\underline{u}_1{}^2 - \underline{v}_1{}^2\right) &= m_2\underline{v}_2{}^2 \end{aligned}$$

Eliminating v_2:

$$\begin{aligned} m_1 m_2\left(\underline{u}_1{}^2 - \underline{v}_1{}^2\right) &= m_1{}^2\left(\underline{u}_1{}^2 - \underline{v}_1{}^2\right)^2 \\ &= m_1{}^2\left(u_1{}^2 + v_1{}^2 - 2u_1 v_1\cos\theta\right)^2 \\ \cos(\theta) &= \frac{u_1}{2v_1}\left(1 - \frac{m_2}{m_1}\right) + \frac{v_1}{2u_1}\left(1 + \frac{m_2}{m_1}\right) \end{aligned}$$

Differentiating with respect to v_1:

$$\begin{aligned} -\frac{u_1}{2v_1{}^2}\left(1 - \frac{m_2}{m_1}\right) + \frac{1}{2u_1}\left(1 + \frac{m_2}{m_1}\right) &= 0 \\ \rightarrow v_1 &= u_1\sqrt{\frac{1 - \frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}}} \end{aligned}$$

Hence, the maximum deflection angle for the larger mass is given by:

$$\theta = \sin^{-1}\left(\frac{m_2}{m_1}\right) \tag{1.10}$$

1.3 Resisted Motion

Most of the scenarios we deal with will not include resistive forces, as this evidently makes the situation much more complicated. However, a particularly important case to take account of such restive forces is with motion under gravity, such as for falling objects. This means that we can introduce the concept of the *terminal velocity* of an object; the object will reach a stage where the resistive force is equal an opposite to the downwards gravitational force, and so the object will stop accelerating. In general, terminal velocity (v_T) occurs as $t \to \infty$ where $\frac{dv}{dt} = 0$.

There are two main types of resistive forces in this case:

- Laminar flow - This is where the resistive force is $\propto v$. It mainly acts when the body is at lower speeds
- Turbulent flow - This is where the resistive force is $\propto v^2$. It begins to act when the body achieves higher speeds

In most cases however, we shall only deal with one of these forces at a time. Recall that:

$$\frac{d\underline{v}}{dt} = \frac{d\underline{x}}{dt} \cdot \frac{d\underline{v}}{dx} = v\frac{d\underline{v}}{dx} \tag{1.11}$$

Depending on whether we want a time or distance dependant expression, we can solve the equations of motion by writing the acceleration as one of these expressions.

1.3.1 Resisted Projectile Motion

This is a slightly more mathematically involved example. Suppose that the resistance experiences by a projectile is launched at a speed u at some angle θ to the horizontal, and that is experiences a resistive force $\underline{F}_{res} = -mk\underline{v}$. Then, by (1.1), we can write the equations of motion as:

$$\begin{aligned} m\frac{d\underline{v}}{dt} &= -\underline{F}_{res} + m\underline{g} \\ &= -mk\underline{v} + m\underline{g} \end{aligned}$$

Separating the two equations into components:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -k \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

We can then solve these two separable equations to obtain the solutions of:

$$\begin{aligned} x &= \frac{u\cos(\theta)}{k}\left(1 - e^{-kt}\right) \\ y &= \frac{1}{k}\left(u\sin(\theta) + \frac{g}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t \end{aligned}$$

It has been left an an exercise to the reader to check these. For simplicity, let $u_o = u\cos(\theta)$ and $v_o = u\sin(\theta)$. *What is the max range of the projectile?* First, let us find the equation of path:

$$\begin{aligned} x &= \frac{u_o}{k}\left(1 - e^{-kt}\right) \\ t &= -\frac{1}{k}ln\left(1 - \frac{kx}{u_o}\right) \\ \to y &= \left(v_o + \frac{g}{k}\right)\frac{x}{u_o} + \frac{g}{k^2}ln\left(1 - \frac{kx}{u_o}\right) \end{aligned}$$

Now, for max range we want $y = 0$. We can use the Taylor expansion of the logarithm and ignore terms of $O(k^3)$ and above for small resistive forces:

$$\begin{aligned}
0 &= \frac{v_o}{u_o}x + \frac{g}{ku_o}x + \frac{g}{k^2}\left(-\frac{k}{u_o}x - \frac{1}{2}\frac{k^2}{u_o{}^2}x^2 - \frac{1}{2}\frac{k^3}{u_o{}^2}x^3\right) \\
&= \frac{v_o}{u_o} - \frac{1}{2}\frac{g}{u_o{}^2}x - \frac{1}{3}\frac{gk}{u_o{}^2}x^2 \\
\to x &= \frac{3u_o{}^2}{2gk}\left(-\frac{1}{2}\frac{g}{u_o{}^2} \pm \sqrt{\frac{1}{4}\frac{g^2}{u_o{}^4} + \frac{4}{3}\frac{v_o gk}{u_o{}^4}}\right)
\end{aligned}$$

Then, using a second Taylor expansion, except keeping terms only in k:

$$\begin{aligned}
x &= -\frac{3}{4}\frac{u_o}{k}\left(1 \pm \left(1 + \frac{8}{3}\frac{v_o k}{g} - \frac{1}{8}\left(\frac{16}{3}\right)^2\left(\frac{v_o{}^2k^2}{g^2}\right)^2\right)\right) \\
&= -\frac{3}{4}\frac{u_o}{k}\left(1 - 1 - \frac{8}{3}\frac{v_o k}{g} + \frac{32}{9}\frac{v_o{}^2k^2}{g^2}\right) \\
\to x_{max} &= \frac{2u_o v_o}{g} - \frac{8}{3}\frac{v_o{}^2u_o k}{g^2}
\end{aligned}$$

Evidently, this is just an approximation, but there is no way to analytically solve the problem.

1.4 Variable Mass Problems

The form that (1.1) is written in becomes more relevant when the body that is being considered has a variable mass. This is because the conservation of momentum has to be applied both to the body and the mass that is lost.

For variable mass problems where the mass is being acquired or lost vertically at zero velocity, the net force on the system is just equal to the weight of the mass. For a rate of mass ejection α:

$$\begin{aligned}\sum \underline{F} &= m\frac{d\underline{v}}{dt} \\ &= m_o\underline{g} - (m_o - \alpha t)\underline{g} \\ &= \alpha \underline{g} t\end{aligned}$$

This just represents the loss of momentum from the system that results from the body losing mass.

1.4.1 Rocket Motion

One very common type of problem that is encountered when considering variable mass is that of a rocket, as it evidently it decreases in mass as fuel is forced out of the back of the rocket. We can derive the equation of motion of the rocket my considering the conservation of momentum from first principles. Consider Figure (1.2), and suppose that the fuel is ejected from the rocket at a rate α with relative speed u.

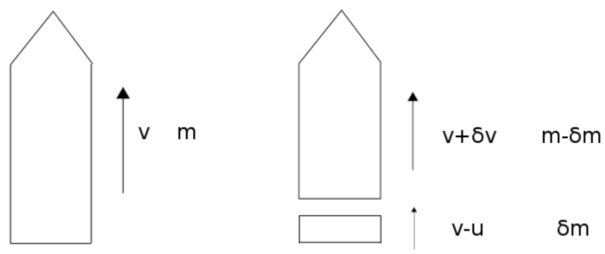

Figure 1.2: Deriving the Rocket Equation

By (1.1):

$$\begin{aligned}\delta \underline{p} &= (m - \delta m)(\underline{v} + \delta \underline{v}) + \delta m(\underline{v} - \underline{u}) - m\underline{v} \\ &= m\delta \underline{v} - \delta m \underline{u} - \delta m \delta \underline{v} \\ &= m\delta \underline{v} - \delta m \underline{u}\end{aligned}$$

as $\delta m \delta \underline{v}$ is very small. Now, $dm = -\delta m$, so dividing through by δt and letting $\delta t \to 0$, we arrive at the *Rocket Equation*:

$$\frac{d\underline{p}}{dt} = m\frac{d\underline{v}}{dt} + \underline{u}\frac{dm}{dt} \tag{1.12}$$

We can then use the fact that $\frac{dm}{dt} = -\alpha$ to find the final velocity of the rocket upon burnout (i.e when all the fuel has been used up).

$$\begin{aligned} t_{burnout} &= \frac{m_{fuel}}{\alpha} \\ -mg &= m\frac{dv}{dt} + u\frac{dm}{dt} \\ &= m\frac{dv}{dt} - u\alpha \\ \frac{dv}{dt} &= -g + \frac{u\alpha}{m_o - \alpha t} \end{aligned}$$

This has solution:

$$\rightarrow v_{burnout} = u \cdot ln\left(\frac{m_o}{m_o - \alpha t_b}\right) - gt_b$$

where $t_b = t_{burnout}$. A couple of points to bear in mind when solving rocket problems:

- To find the height, integrate the expression of the velocity with respect to time up until the burnout
- To find the *max* height, use conservation laws to find the extra height gained due to the velocity that remains upon burnout
- For multi-stage rockets, transform to the frame moving at burnout velocity, and solve the problem of the next stage in this frame. Then, transform back to the initial frame by adding the initial velocity; the final velocity is just the sum of the burnout velocities of each stage assuming that the jettisoning of each stage is instantaneous

1.4.2 Relativistic Rocket

This is an interesting problem that involved relativistic variable mass. It is recommended that the reader takes a look at Chapter 3 before reading through this problem.

Suppose we have a rocket that converts mass directly into energy and ejects it from the back of the rocket in the form of photons. Initially, let us consider the problem in the rest frame of the rocket. By the conservation of momentum:

$$mdv' + v\gamma dm_f = 0$$

By the conservation of energy:

$$\begin{aligned} dm - \gamma dm_f &= 0 \\ mdv' &= -vdm \\ m\frac{dv'}{dm} + v &= 0 \end{aligned}$$

We now need to transform to the rest frame of the stationary observer. Using (3.9):

$$\begin{aligned} v + dv &= \frac{v + dv'}{1 + v\frac{dv'}{c^2}} \\ &\approx (v + dv')\left(1 - \frac{v}{c^2}dv'\right) \\ &= v + dv' - \frac{v^2}{c^2}dv' - \frac{v}{c^2}dv'^2 \\ &= v + dv' - \frac{v^2}{c^2}dv' \\ \rightarrow dv &= dv'\left(1 - \frac{v^2}{c^2}\right) \end{aligned}$$

Now, as photons travel at $v = c$, we arrive at the equation of motion:

$$m\frac{dv}{dm} + c\left(1 - \frac{v^2}{c^2}\right) = 0$$

This separable equation can then be solved using partial fractions to find the subsequent evolution of the system.

2. *Mechanics II*

This section aims to cover some of the more advanced concepts in Mechanics, including:

- Dimensional Analysis
- Dynamical Coordinate Systems
- Rotational Dynamics
- Central Forces and Orbits
- Lagrangian Mechanics

In all likelihood, most of the concepts covered in this section will be new to most readers. Quite a few examples have been included to help facilitate further familiarity with said concepts. This chapter assumes that readers are already very competent with using the concepts elucidated in Chapter (1).

2.1 Dimensional Analysis

Dimensional analysis is a useful tool for physics in general; it can be used to find the form of physical formulae, or even check results obtained as a solution to a problem. A *dimension* is a fundamental quality of a physical quantity, and can include things such as length, time, mass and electric charge. Every other physical unit can be derived from some combination of these dimensions. This means that, say, we expect an answer to have units of m, then if the result does not have dimensions of length, then we know that something has gone awry. Note that when we input numbers into an expression the dimensions do not simply 'disappear'; numbers still have dimensions, unless of course that number is meant to be dimensionless.

We denote the dimensions of some quantity Q by the notation $[Q]$. The 'fundamental' dimensions are:

$$\begin{aligned} [Time] &= T \\ [Metres] &= L \\ [Mass] &= M \\ [Charge] &= Q \end{aligned}$$

The dimensions of some common physical units include:

$$\begin{aligned} [Force] &= MLT^{-2} \\ [Energy] &= ML^2T^{-2} \\ [Power] &= ML^2T^{-3} \\ [Current] &= QT^{-1} \\ [EMF] &= ML^2T^{-2}Q^{-1} \end{aligned}$$

We can also use dimensional analysis to derive formulae to within some dimensionless constant. The general method is to write the desired quantity as a product of the quantities it depends on to a set of powers, and then equate the powers of the resultant dimensional quantities. For example, suppose we want to obtain the formula for the cyclotron frequency. We know that $\Omega \propto B,\ m\ ,\ q$. Thus:

$$\begin{aligned} \Omega &= k \cdot B^a \cdot m^b \cdot q^c \\ [\Omega] &= [k] \cdot [B]^a \cdot [m]^b \cdot [q]^c \\ T^{-1} &= (MT^{-1}Q^{-1})^a \cdot (M)^b \cdot (Q)^c \\ &= M^{a+b} \cdot T^{-a} \cdot Q^{c-a} \end{aligned}$$

Equating powers on either side of the equation, it becomes clear that:

$$\begin{aligned} a + b &= 0 \\ a &= 1 \\ c - a &= 0 \\ \rightarrow\ a &= 1\ ,\ b = -1\ ,\ c = 1 \end{aligned}$$

The cyclotron frequency is thus given by:

$$\Omega = k \cdot \frac{qB}{m}$$

2.2 Dynamical Coordinate Systems

Generally, we are used to coordinate systems being time-independent; the x-axis remains in the same direction throughout all time, as is the case with all of the Cartesian coordinate axes. However, some coordinate systems are dynamical, meaning that they change with time. One such coordinate system is the polar coordinate system with unit vectors $\underline{\widehat{r}}$ and $\underline{\widehat{\theta}}$. Recall that $\underline{\widehat{r}} \cdot \underline{\widehat{r}} = 1$.

$$\begin{aligned} \frac{d}{dt}\left(\underline{\widehat{r}} \cdot \underline{\widehat{r}}\right) &= 0 \\ \underline{\widehat{r}} \cdot \frac{d\underline{\widehat{r}}}{dt} &= 0 \\ \rightarrow \underline{\widehat{r}} \perp \frac{d\underline{\widehat{r}}}{dt} & \end{aligned}$$

This means that the time derivative of $\underline{\widehat{r}}$ is not in the same direction as itself. In fact,

$$\dot{\underline{\widehat{r}}} = \dot{\theta} \cdot \underline{\widehat{\theta}} \tag{2.1}$$

$$\dot{\underline{\widehat{\theta}}} = -\dot{\theta} \cdot \underline{\widehat{r}} \tag{2.2}$$

This means that the acceleration in polar coordinates is written as:

$$\ddot{\underline{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{\widehat{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{\widehat{\theta}} \tag{2.3}$$

This follows from the differentiation of the position vector $\underline{r}$.

2.2.1 Rotating Reference Frame

For some problems, it helps to work in a rotating reference from. For $\underline{F} = m\ddot{\underline{r}}$, we can write $\underline{F'}$ in the rotating frame as:

$$m\ddot{\underline{r'}} = F - m\dot{\underline{\omega}} \times \underline{r'} - 2m\underline{\omega} \times \dot{\underline{r'}} + m\underline{\omega} \times \left(\underline{\omega} \times \underline{r'}\right) \tag{2.4}$$

Note that in most cases, $\underline{r'} \parallel \underline{r}$, but the magnitude is not necessarily the same. The second and third terms on the right-hand side of the equation represent the Coriolis force, and the fourth represents the Centripetal force.

Consider the example of firing a projectile from a tower of height h on the equator of the Earth. Let the Earth have angular velocity ω. The distance taken for the projectile to reach the ground is most easily found by considering (2.4). We know that $\dot{\underline{\omega}} = 0$ as the speed of rotation of the earth is constant, and we can ignore $\underline{\omega} \times (\underline{\omega} \times \underline{r'})$ as $\underline{\omega}$ is small.

$$\begin{aligned} m\ddot{\underline{r'}} &= F - 2m\underline{\omega} \times \dot{\underline{r'}} \\ \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} &= -\begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \\ &= -\begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix} - 2\omega\begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} \end{aligned}$$

In comparison to $\dot{y}$, $\dot{x}$ is very small.

$$\begin{aligned}\ddot{y} &= -g + 2\omega\dot{x} \\ &\approx -g \\ \dot{y} &= -gt \\ y &= -\frac{1}{2}gt^2 \\ \rightarrow t_{fall} &= \sqrt{\frac{2h}{g}}\end{aligned}$$

However, the other equation of motion gives:

$$\begin{aligned}\ddot{x} &= -2\omega\dot{y} \\ &= 2\omega gt \\ x &= \frac{1}{3}\omega gt^3\end{aligned}$$

Hence, the distance that the projectile ends up from the base of the tower is:

$$x = \frac{2}{3}\omega h \cdot \sqrt{\frac{2h}{g}}$$

This is assuming that the radius of curvature of the earth is sufficiently large that the surface of the earth can be considered to be locally flat.

2.3 Rotational Dynamics

This section focusses on the concept of rotation. Generally, with think of the motion of an object in mechanics as simple linear motion of the object. However, during it's motion, the object may have some rotation about the centre of mass. In fact, as we will see, any motion of the system can be split into the translation of the centre of mass, and a rotation about the centre of mass.

2.3.1 Angular Momentum and Torque

Angular momentum is defined as:

$$\underline{L} = \underline{r} \times \underline{p} \tag{2.5}$$

Under the condition that $\underline{r} \perp \underline{p}$, this can also be written as:

$$\begin{aligned} L &= I\omega \\ &= mvr \\ &= mr^2\dot{\theta} \end{aligned}$$

From the cross product, it becomes clear that the direction of angular momentum is perpendicular to the plane of the instantaneous motion.

$$\begin{aligned} \frac{d\underline{L}}{dt} &= \underline{r} \times \underline{F} + \underline{\dot{r}} \times \underline{p} \\ &= \underline{r} \times \underline{F} + \underline{\dot{r}} \times m\underline{\dot{r}} \\ &= \underline{r} \times \underline{F} \end{aligned}$$

We defined the torque on an object as:

$$\underline{\tau} = \underline{r} \times \underline{F} = \frac{d\underline{L}}{dt} \tag{2.6}$$

Thus, *angular momentum is conserved in the absence of external torques.* This is the 'rotational analogue' (if you like) of (1.1). Let us consider the total angular momentum of the system using the centre of mass quantities:

$$\begin{aligned} \underline{L} &= \sum_i \underline{L}_i \\ &= \sum_i m_i \cdot (\underline{r}'_i + \underline{r}_{cm}) \times (\underline{v}_i + \underline{v}_{cm}) \\ &= \sum_i m_i \cdot (\underline{r}'_i \times \underline{v}'_i) + \sum_i m_i \cdot (\underline{r}'_i \times \underline{v}_{cm}) + \sum_i m_i \cdot (\underline{r}_{cm} \times \underline{v}'_i) + \sum_i m_i \cdot (\underline{r}_{cm} \times \underline{v}_{cm}) \end{aligned}$$

The first and third terms will disappear by the definition of the centre of mass. Hence, we obtain:

$$\underline{L} = \underline{L}' + \underline{r}_{cm} \times M\underline{v}_{cm} \tag{2.7}$$

As previously stated, this means we can characterise motion as the translation of the CM and a rotation around it.

2.3.2 Moment of Inertia

There is a quantity used a lot in rotational mechanics called the *moment of inertia* of a body. It is defined as:

$$I = \int r^2 \cdot dm \tag{2.8}$$

where $dm = \rho \cdot dV$, and r is the perpendicular distance to the axis of rotation. If we wish to work out rotational quantities about a particular axis, we first need to calculate the moment of inertia about said axis.

The moments of inertia of some common objects are as follows:

- Thin rectangular plate of side lengths a and b:

$$\begin{aligned} dm &= \frac{M}{ab} \cdot dxdy \\ I_x &= \frac{1}{12}Mb^2 \\ I_y &= \frac{1}{12}Ma^2 \\ I_z &= \frac{1}{12}M\left(a^2+b^2\right) \end{aligned}$$

- Thin circular disk of radius R:

$$\begin{aligned} dm &= \frac{M}{\pi R^2} \cdot (2\pi r \cdot dr) \\ \to I_z &= \frac{1}{2}MR^2 \end{aligned}$$

- A sphere of radius R. We want to calculate this by summing up the moments of inertia of the disks that make up the sphere:

$$\begin{aligned} dI &= \frac{1}{2}x^2 dm \\ dm &= (\pi x^2)\rho dz \\ \rho &= \frac{3M}{4\pi R^3} \\ x^2 &= R^2 - z^2 \\ \to I &= \frac{2}{5}MR^2 \end{aligned}$$

- Long, narrow rod of length l:

$$\begin{aligned} dm &= \frac{M}{l} \\ \to I_z &= \frac{1}{12}Ml^2 \end{aligned}$$

- Solid Cylinder of radius R and length l:

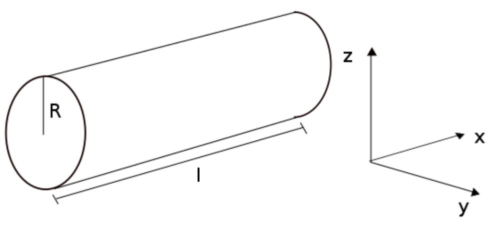

Figure 2.1: Coordinate system for deriving the Moment of Inertia of a Cylinder

– I_x :

$$dm = \frac{M}{\pi R^2 l} \cdot (2\pi r \cdot dr dx)$$
$$\rightarrow I_x = \frac{1}{2} M R^2$$

– I_y :

$$dI_x = \frac{1}{2} dm R^2$$
$$= 2 dI_y$$
$$dI_z = \frac{1}{4} dm R^2 + \frac{1}{4} dm x^2$$
$$\rightarrow I_z = \frac{1}{4} M R^2 + \frac{1}{12} M L^2$$

Some integration steps have not been included; it is recommended that the reader works through each of these examples on their own to get a handle on deriving moments of inertia.

2.3.3 Axis Theorems

There are two 'axis' theorems that can be used to simplify or manipulate moment of inertia calculations. These are as follows:

- Parallel Axis Theorem:

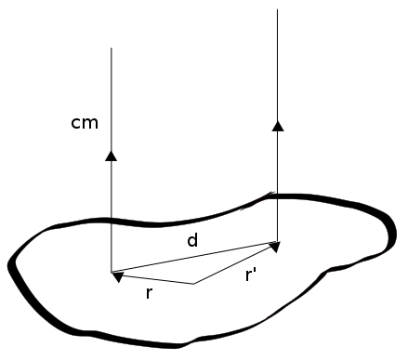

Figure 2.2: Deriving the Parallel Axis Theorem

Let $\underline{d}$ be a constant vector that gives the position of the parallel axis relative to the axis through the centre of mass.

$$\underline{r}' = \underline{r} + \underline{d}$$
$$I = \int \underline{r}'^2 \cdot dm$$
$$= \int r^2 \cdot dm + d^2 \cdot \int dm + 2\underline{d} \cdot \int \underline{r} dm$$

However, the third term will again disappear by the definition of the centre of mass. Hence we obtain the *Parallel Axis Theorem*:

$$I = I_{cm} + M d^2 \tag{2.9}$$

- Perpendicular Axis theorem:

$$\begin{aligned}
I_z &= \int d^2 \cdot dm \\
&= \int (x^2 + y^2) \cdot dm \\
&\rightarrow I_z = I_x + I_y
\end{aligned} \tag{2.10}$$

  Note that this only holds true for a planar body. One consequence of this theorem is if the body is uniform (e.g a regular pentagonal laminar), *the moment of inertia through any two orthogonal axes must be the same.*

2.3.4 Example Problems

Below are a range of problems covering different aspects of the material in this section. They should serve as examples that can be referenced when working on harder or more involved problems.

1. Imagine that a ball of radius a rolls without slipping down a ramp inclined at an angle θ to the horizontal. Let x be the distance moved by the ball down the ramp, and h be the vertical distance of the ball below the top of the ramp. *What is the acceleration of the ball down the ramp?*

   Firstly, the condition for a ball to roll without slipping is given by $v = \omega r$, as the edge has to move at the same velocity as the CM. Furthermore, if the ball rolls without slipping, we know that no energy is dissipated by frictional forces. This means that energy is conserved. We can thus write:

$$\begin{aligned}
mgh &= \frac{1}{2}m\left(v^2 + I\omega^2\right) \\
&= \frac{1}{2}m\left(v^2 + \frac{I}{ma^2}v^2\right) \\
&= \frac{1}{2}mv^2(k+1) \\
for\ \ k &= \frac{I}{ma^2}
\end{aligned}$$

   Re-arranging and substituting for h:

$$\begin{aligned}
v^2 &= \frac{2gh}{k+1} \\
&= \frac{2gx\sin(\theta)}{k+1}
\end{aligned}$$

   Differentiating with respect to time:

$$\begin{aligned}
2v \cdot \frac{dv}{dt} &= \frac{2g\sin(\theta)}{k+1} \cdot v \\
\rightarrow \frac{dv}{dt} &= \frac{g\sin(\theta)}{k+1}
\end{aligned}$$

   Evidently, it is clear that does not only hold for a sphere sphere, but in fact holds for any object able to roll without slipping.

2. Imagine that a snooker ball of radius a sits on a rough, horizontal table. *At what height does one have to hit the ball with the cue such that it moves off without slipping?*

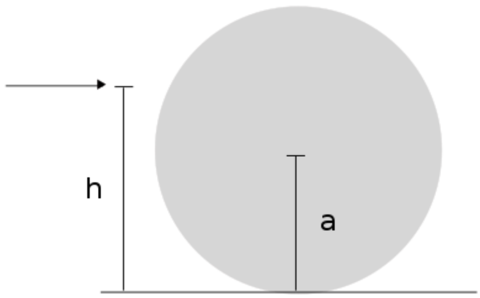

Figure 2.3: Hitting a snooker ball

Equating torques around the ball's CM:

$$F(h-a) = I_{cm} \cdot \dot{\omega}$$

For rolling without slipping:

$$\dot{\omega} = \frac{\dot{v}}{a}$$
$$F(h-a) = I_{cm} \cdot \frac{\dot{v}}{a}$$

However, by (1.1):

$$\dot{v} = \frac{F}{m}$$
$$h - a = \frac{I_{cm}}{ma}$$
$$\rightarrow h = \frac{I_{cm}}{ma} + a$$

For a sphere, this becomes:

$$h = \frac{7a}{5}$$

Now consider what happens if we do not hit the ball in this "sweet spot". *How far does it travel before rolling without slipping begins?* Again, let us equate torques around the centre of mass:

$$F \cdot r = I \cdot \dot{\omega}$$
$$F = \mu mg \;\rightarrow\; \dot{\omega} = \frac{\mu mgr}{I}$$
$$\omega = \frac{\mu mgrt}{I}$$
$$v = \frac{\mu mgr^2 t}{g}$$

We have imposed the rolling without slipping condition. The centre of mass acceleration is given by $\ddot{r} = -\mu g$.

$$\begin{aligned} \frac{\mu m g r^2 t}{I} &= v_o - \mu g t \\ t\left(\mu g + \frac{\mu m g r^2}{I}\right) &= v_o \\ \rightarrow t_o &= v_o \cdot \left(\mu g + \frac{\mu m g r^2}{I}\right)^{-1} \end{aligned}$$

Thus, the distance travelled by the CM is:

$$\begin{aligned} d &= v_o \cdot t - \frac{1}{2} \cdot \ddot{r} \cdot t^2 \\ &= v_o \cdot t_o - \frac{1}{2} \cdot (\mu m g) \cdot {t_o}^2 \end{aligned}$$

And then solve the equation numerically for d.

3. Imagine that a rod of mass m and length l is released horizontally under gravity without resistive forces. An force of magnitude F is delivered instantaneously to one end of the rod that causes the rod to rotate with angular frequency ω_o. *What magnitude of force is required such that the rod returns exactly to its starting position?*

The subsequent motion of the rod involve the motion of the centre of mass upwards and a rotation around the centre of mass, as shown in Figure (2.4).

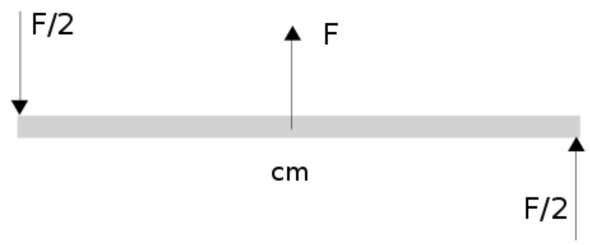

Figure 2.4: Decomposing the vertical impulse on a rod

We want to find the time to max height by considering the change in angular momentum on the rod:

$$\begin{aligned} F\delta t \frac{l}{2} &= I\omega_o \\ F\delta t &= m v_o \\ \frac{l}{2} m v_o &= I\omega_o \\ v_o &= \frac{\omega_o l}{6} \\ \rightarrow t_{\,max\ height} &= \frac{\omega_o l}{6g} \end{aligned}$$

Using the fact that $I = \frac{1}{12} m l^2$ about the CM for a rod. Now, for it to return exactly to it's starting position, we require it to execute an integer number of full revolutions in the time it takes to rise and then fall again. Mathematically, we can express this as:

$$2 \cdot \left(\frac{\omega_o l}{6g}\right) \omega_o = 2\pi n$$
$$\omega_o{}^2 = \frac{6\pi n g}{l}$$
$$F = \frac{2I\omega_o}{l}$$
$$= \frac{1}{6} m l \omega_o$$
$$\rightarrow F = \frac{1}{6} m \sqrt{6\pi n g l}$$

Thus, there are actually an infinite number of forces that can be applied as n can take any integer value; the only thing that changes is the maximum height and subsequently the number of rotations.

4. Consider a rod of length l and mass m lying on a flat horizontal table. Suppose that an impulse is delivered at a distance x from one end of the rod. *What is the value of x that is required such that the end of the rod experiences no initial acceleration?* This is a typical 'centre of percussion' problem.

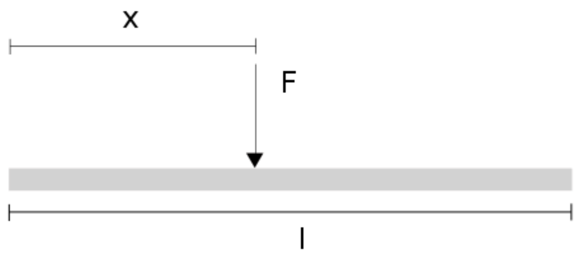

Figure 2.5: The centre of percussion for a rod

Finding the angular momentum about the end of the rod:

$$F \cdot x = I_{end} \cdot \omega$$

The linear acceleration of the CM is given by:

$$F = m\dot{v}$$
$$\dot{v} = \frac{F}{m}$$

The linear acceleration of the end of the rod is given by:

$$\dot{v} = \frac{Fx}{I_{end}} \cdot \frac{l}{2}$$

For the end of the rod to experience a net zero acceleration, this must be equal to the linear acceleration of the CM.

$$\frac{F}{m} = \frac{Fx}{I_{end}} \cdot \frac{l}{2}$$
$$\rightarrow x = \frac{2}{3} l$$

using the fact that the moment of inertia of a rod about its end is $I_{end} = \frac{1}{3} m l^2$.

2.4 Central Forces and Orbits

This section is all about tackling problems involving the orbital motion of bodies. Note that in all cases, we will only consider the motion of a body under a single central force, or at least a system that can be modelled as equivalent to a single central force.

What is a central force? Well, a central force is one that only acts in the radial direction, i.e $\underline{F} = F(r)\underline{\hat{r}}$. Readers will already be familiar with a number of central forces, such as the force due to Coulomb's Law or *Newton's Law of Universal Gravitation*:

$$\underline{F}_g = \frac{Gm_1m_2}{r^2} \cdot \underline{\hat{r}} \tag{2.11}$$

By it's definition, a central force will always pull an object inwards toward the origin of the central force, and the motion of objects under the influence of these central forces is what we typically think of as orbits. For a very simple circular orbit, the central force is analogous to the centripetal force that keeps the object in circular orbit.

Central forces have two main properties when applied to orbital mechanics:

- Under a central force, angular momentum is conserved

$$\begin{aligned}
\underline{L} &= \underline{r} \times \underline{p} \\
\frac{d\underline{L}}{dt} &= \underline{r} \times \underline{\dot{p}} + \underline{\dot{r}} \times \underline{p} \\
&= \underline{r} \times \underline{\dot{p}} + \underline{\dot{r}} \times m\underline{\dot{r}} \\
&= \underline{r} \times \underline{F}
\end{aligned}$$

  Using the fact that for a central force $\underline{F} = F(r)\underline{\hat{r}}$:

$$\rightarrow \frac{d\underline{L}}{dt} = \underline{r} \times F(r)\underline{\hat{r}} = 0 \tag{2.12}$$

- Under a central force, the motion is confined to a plane. We know that the momentum of a body is given by $\underline{L} = \underline{r} \times \underline{p} = m(\underline{r} \times \underline{\dot{r}})$. It is clear that $\underline{r} \cdot \underline{L} = 0$. As $\underline{L}$ is a constant vector from (2.12), this is an equation of a plane through the origin. Hence the motion is confined to a plane.

2.4.1 Kepler's Laws

Kepler's Laws govern the motion of orbital bodies, though they are less useful tools for solving a lot of orbital problems. They are as follows:

1. The orbit of every plane is an ellipse with the sun at one of the foci
2. A line joining the planet and the sun sweeps out equal areas in equal times

$$\begin{aligned}
dA &= \frac{1}{2}r(rd\theta) \\
&= \frac{1}{2}r^2 d\theta \\
\frac{dA}{dt} &= \frac{1}{2}r^2\frac{d\theta}{dt} \\
&= \frac{L}{2m} \\
\rightarrow \frac{dA}{dt} &= constant
\end{aligned}$$

3. For a body orbiting a larger body of mass M with radius of orbit r and period T:
$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2} \tag{2.13}$$
This can be derived by equating the gravitational (2.11) and centripetal forces for an object performing circular motion.
$$\frac{mv^2}{r} = \frac{GmM}{r^2}$$
For a circle,
$$v = \frac{2\pi r}{T}$$
Substituting this result in:
$$\begin{aligned} \frac{4\pi^2 r^2}{rT^2} &= \frac{GM}{r^2} \\ \rightarrow \frac{r^3}{T^2} &= \frac{GM}{4\pi^2} \end{aligned}$$
Despite deriving this equation assuming a circular orbit, it does actually hold for any periodic orbit (*not* hyperbolic orbits), though the proof for this is significantly more complicated.

2.4.2 Energy of Orbital Motion

Most problems in orbital mechanics are most easily solved by considering the total energy of the object. As angular momentum is conserved under central forces, energy is also conserved assuming no impulse is given to the body. We know that:
$$E_{total} = KE + U$$
Let us first consider the kinetic energy:
$$\begin{aligned} \underline{v} &= \begin{pmatrix} \dot{r} \\ r\dot{\theta} \end{pmatrix} \\ \underline{v}^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\ KE &= \frac{1}{2}m\underline{v}^2 \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \end{aligned}$$
By the definition of angular momentum for the case where $\underline{r} \perp \underline{p}$:
$$\begin{aligned} L &= mvr \\ &= mr^2\dot{\theta} \\ \rightarrow KE &= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} \end{aligned}$$
Now, the potential energy is just the work done against the central force to bring the object in from the zero of potential, that we will define as being at ∞.
$$\begin{aligned} U &= -\int \underline{F} \cdot d\underline{r} \\ &= -\int_r^\infty \frac{GmM}{r^2} \cdot dr \\ \rightarrow U &= -\frac{GmM}{r} \end{aligned}$$

Note that this assumes that $M \gg m$, as we have assumed the centre of mass of the two bodies is located at the centre of the body of mass M. If not, we have to use the Reduced Mass System, as in Section (2.4.4).

The total energy of the system is thus given by the Orbit Equation:

$$E_{total} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} \tag{2.14}$$

2.4.3 The Effective Potential

The quantity:

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{GmM}{r}$$

is known as the *Effective Potential* of the system; it is the potential 'felt' by the object when the system has been reduced to a single variable equation (in this case r). It is an analytical tool that can be used to examine the effect that changes in L and $\dot{r}$ have on the system, as well as for classifying the type of orbit.

Graphically, we represent the effective potential as:

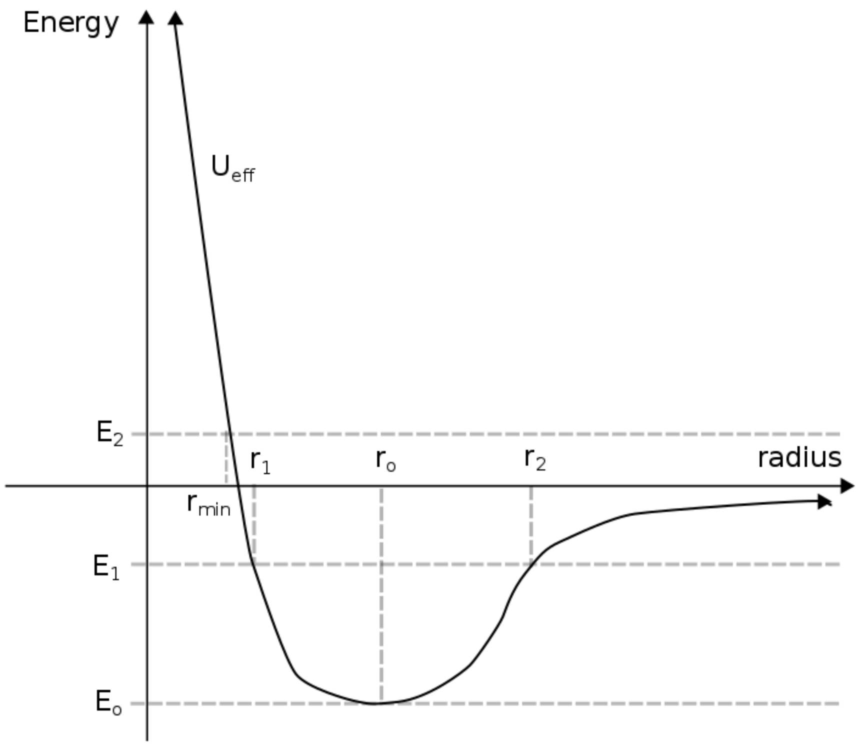

Figure 2.6: The Effective Potential

We have three distinct cases for the type of orbit based on the value of the energy:

- E_o - Circular motion with $r = r_o$
- E_1 - Elliptical motion with $r_1 < r < r_2$

- E_2 - Unbounded or Hyperbolic motion with $r > r_{min}$

For an impulse delivered to an orbiting body, this can have an effect on the orbit in one of two ways, depending on the direction in which the impulse is applied:

- Tangentially, $L \longrightarrow L + \Delta L$ - This causes a shift in the effective potential curve on the graph (though the total energy remains the same), causing the radii to change.
- Radially, $\dot{r} \longrightarrow \dot{r} + \Delta\dot{r}$ - This causes a shift in the total energy, but the effective potential curve remains the same. For example, the energy may shift from E_o to E_1.

When in doubt, always sketch the effective potential; seeing the energy represented graphically can sometimes be the key to solving a tricky problem!

2.4.4 The Reduced Mass System

As previously stated, the Orbit Equation (2.14) is not valid when the two bodies are of comparable size. In this case, we can treat it using the Reduced Mass System. Consider Figure (2.7).

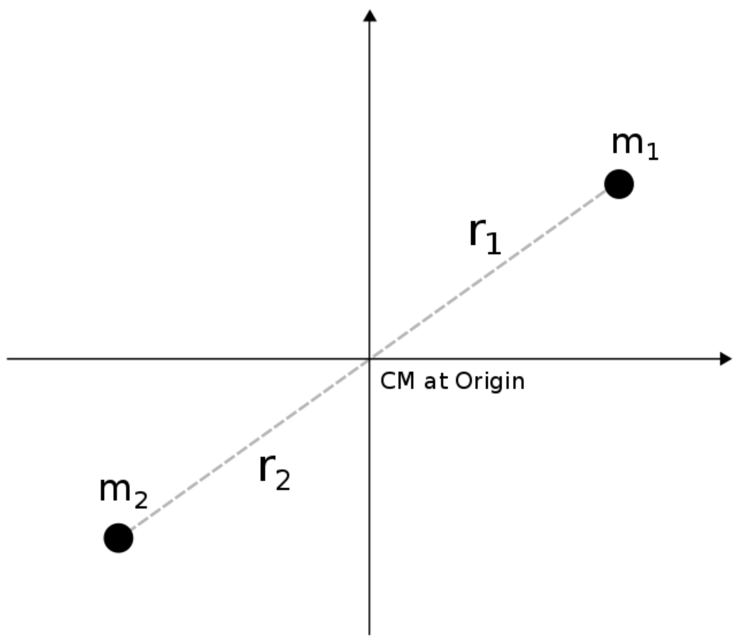

Figure 2.7: The Reduced Mass System

By (1.1),

$$\begin{aligned} m_1 \cdot \ddot{\underline{r}}_1 &= \underline{F}_{12} \\ m_2 \cdot \ddot{\underline{r}}_2 &= \underline{F}_{21} = -\underline{F}_{12} \\ \ddot{\underline{r}}_1 &= \frac{\underline{F}_{12}}{m_1} \\ \ddot{\underline{r}}_2 &= -\frac{\underline{F}_{12}}{m_2} \end{aligned}$$

Let $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1+m_2}$ (the reduced mass). Then:

$$\begin{aligned} \ddot{\underline{r}} &= \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 \\ &= -\frac{1}{\mu} \cdot \underline{F}_{12} \\ \rightarrow \mu \cdot \ddot{\underline{r}} &= -\frac{G\mu M}{r^2} \cdot \widehat{\underline{r}} \end{aligned}$$

We can use this equation of motion to find the period of the orbit of the system.

$$\begin{aligned} \ddot{\underline{r}} + \frac{GM}{r^3} \cdot \underline{r} = 0 \\ \rightarrow T = 2\pi \sqrt{\frac{r^3}{G(m_1+m_2)}} \end{aligned}$$

Note that this reduces to circular motion if $m_1 = m_2$, as this implies that $r_1 = r_2$.

2.4.5 Central Force Scattering

If an object located (initially) at the zero of potential energy is given a small velocity v_o, it will be attracted towards the body providing the central force. In this case, it will undergo a hyperbolic orbit, which has two main characteristics:

- Distance of minimum approach - Initially, the body only has energy equal to its angular momentum $E_o = \frac{1}{2} m {v_o}^2$. At the distance of closest approach, the energy is no longer radial, meaning that $\dot{r} = 0$. Assuming energy is conserved:

$$\begin{aligned} E_o &= \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} \\ \frac{1}{2} m {v_o}^2 &= \frac{L^2}{2mr^2} - \frac{GmM}{r} \end{aligned}$$

  As angular momentum is also conserved, we know that $L = mbv_o$ for some characteristic impact parameter b. Thus, solving for r, we obtain:

$$r_{\,min} = -\frac{GM}{{v_o}^2} + \sqrt{\frac{G^2M^2}{{v_o}^4} + b^2} \tag{2.15}$$

- Scattering Angle - Consider Figure (2.8). Evidently, the $\widehat{\underline{x}}$ components of Δp will cancel one another, and so we just have to calculate the $\widehat{\underline{y}}$ components.

$$\begin{aligned} F_y &= \frac{GmM}{r^2} \cdot \cos\theta \\ L &= mr^2 \frac{d\theta}{dt} \;\rightarrow\; \frac{L}{mr^2} \cdot dt = d\theta \end{aligned}$$

  We can use this result to simplify the integral significantly.

$$\begin{aligned} \Delta p &= \int F_y \cdot dt \\ &= \int \frac{GmM}{r^2} \cdot dt \\ &= \frac{Gm^2M}{L} \cdot \int_{-\frac{\pi}{2}-\frac{\varphi}{2}}^{\frac{\pi}{2}+\frac{\varphi}{2}} \cos\theta \cdot d\theta \\ &= 2 \cdot \frac{Gm^2M}{L} \cdot \cos\left(\frac{\varphi}{2}\right) \end{aligned}$$

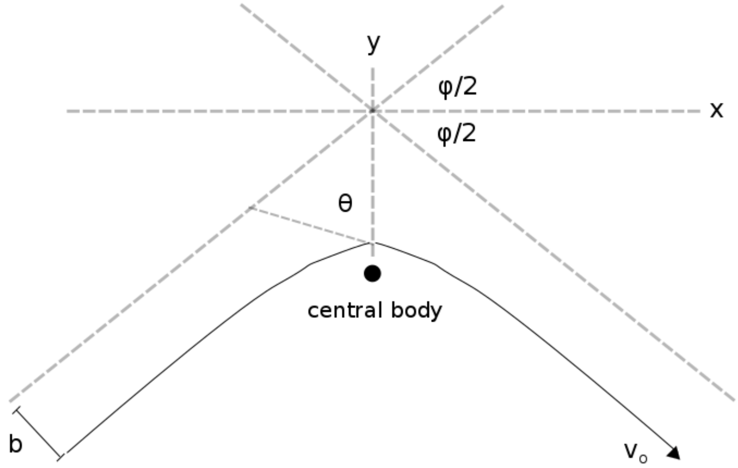

Figure 2.8: The Scattering Angle for a hyperbolic orbit

We now need another expression for Δp. Again looking at Figure (2.8), it becomes clear that this is equal to:

$$\Delta p = 2 \cdot m v_o \cdot \sin\left(\frac{\varphi}{2}\right)$$

Equating these two expressions, we arrive at the final expression for the scattering angle of:

$$\cot\left(\frac{\varphi}{2}\right) = \frac{L v_o}{GmM} = \frac{b {v_o}^2}{GM} \tag{2.16}$$

Note that this is independent of the mass of the body.

2.4.6 Example Problems

Below are a range of problems covering different aspects of the material in this section. They should serve as examples that can be referenced when working on harder or more involved problems.

1. Suppose that NASA scientists wanted to put a satellite into a circular orbit of radius r_o with a velocity v_o, but instead accidentally directed the impulse at an angle θ to the tangent of the desired orbit. *What are the maximum and minimum radii of the new orbit in terms of r_o?*

   In this case, the total energy of the orbit is the same, but the angular momentum is changed. Initially, let us equate the gravitational and centripetal forces for circular motion:

$$\frac{m{v_o}^2}{r_o} = \frac{GmM}{{r_o}^2}$$
$$\rightarrow {v_o}^2 = \frac{GM}{r_o}$$

   This is quite a useful result for simplifying expressions involving circular orbits, so try to remember it!

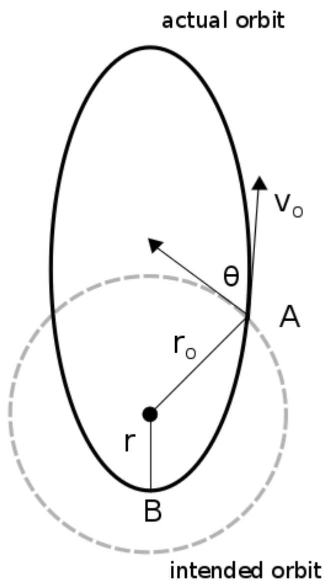

Figure 2.9: The Misdirected Launch! NASA is at it again....

The angular momentum is given by:

$$\begin{aligned} L &= mr_o v_{perp} \\ &= mr_o v_o \cos\theta \end{aligned}$$

As we know the energy in each orbit is the same, we need to find expressions for the energy at points A and B.

$$\begin{aligned} E_A &= \frac{1}{2}m{v_o}^2 - \frac{GmM}{r_o} \\ &= -\frac{1}{2}\frac{GmM}{r_o} \\ E_B &= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} \end{aligned}$$

Equating these two expressions and letting $\dot{r} = 0$:

$$\begin{aligned} -\frac{1}{2}\frac{GmM}{r_o} &= \frac{L^2}{2mr^2} - \frac{GmM}{r} \\ 0 &= \frac{1}{2}\frac{GmM}{r_o}r^2 - (GmM)r + \frac{L^2}{2m} \\ &= r^2 - (2r_o)r + {r_o}^2\cos^2\theta \end{aligned}$$

using the results for L and ${v_o}^2$. Thus, we arrive at the quite simple expression of:

$$\rightarrow r = r_o(1 \pm \sin\theta)$$

2. A comet in a circular orbit around the Sun has speed v_o and radius $r_0 = \alpha R$, where R is the radius of the Earth's orbit, and α is a constant satisfying $0 < \alpha < 1$. The comet has its velocity reduced by Δv in a collision that does not change its initial direction. *What is the minimum Δv required to move the comet into an orbit which intersects that of the Earth?*

The initial energy of the comet is:

$$E_o = \frac{L^2}{2mr_o{}^2} - \frac{GmM}{r_o}$$
$$L = mr_o(v_o - \Delta v)$$

At the instant where it's velocity is reduced, it is still in circular motion. We require that $E_o = E_R$.

$$\frac{L^2}{2mr_o{}^2} - \frac{GmM}{r_o} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mR^2} - \frac{GmM}{R}$$
$$L^2\left(\frac{1}{2mr_o{}^2} - \frac{1}{2mR^2}\right) = \frac{GmM}{r_o} - \frac{GmM}{R}$$
$$L^2 = 2Gm^2M\left(\frac{r_oR}{r_o + R}\right)$$
$$L = \sqrt{2Gm^2M\left(\frac{\alpha R}{1+\alpha}\right)}$$

Substituting $L = mr_o(v_o - \Delta v)$:

$$mr_o(v_o - \Delta v) = \sqrt{2Gm^2M\left(\frac{\alpha R}{1+\alpha}\right)}$$
$$v_o - \Delta v = \sqrt{\frac{2GM}{\alpha^2R^2}\cdot\left(\frac{\alpha R}{1+\alpha}\right)}$$
$$= \sqrt{\frac{GM}{r_o}}\cdot\sqrt{\frac{2}{1+\alpha}}$$
$$\to \Delta v = v_o\left(1 - \sqrt{\frac{2}{1+\alpha}}\right)$$

How does the expression change if the impulse Δv is delivered in the radial direction towards the sun?

In this case, the initial energy of the comet is:

$$E_o = \frac{1}{2}m\Delta v^2 + \frac{L^2}{2mr_o{}^2} - \frac{GmM}{r_o}$$
$$L = mr_ov_o$$

Again, we require that $E_o = E_R$. Using the fact that $R = \frac{r_o}{\alpha}$:

$$\frac{1}{2}m\Delta v^2 + \frac{L^2}{2mr_o{}^2} - \frac{GmM}{r_o} = \frac{L^2}{2mR^2} - \frac{GmM}{R}$$
$$\frac{1}{2}m\Delta v^2 - \frac{1}{2}mv_o{}^2 = \frac{L^2\alpha}{2mr_o{}^2} - \frac{GmM\alpha}{r_o}$$
$$\frac{1}{2}m\Delta v^2 = \frac{1}{2}mv_o{}^2\cdot(\alpha^2+1) - m\alpha v_o{}^2$$
$$= \frac{1}{2}mv_o{}^2(\alpha^2 - 2\alpha + 1)$$
$$\to \Delta v = v_o(\alpha - 1)$$

Notice how the results are markedly different depending on the way in which the impulse is applied. In general, it is much more 'energy-efficient' to transfer between orbits by changing the angular momentum, rather than giving the body some radial velocity.

3. A probe of mass m is initially in circular orbit with radius r_1. *Show that with two appropriately placed impulses, the probe is able to reach a second circular orbit of radius $r_2 > r_1$.*

This question introduced the concept of a Hohmann transfer orbit; it is one of the lowest energy methods for transferring an object between two circular orbits.

Before we tackle this question, we need to derive a useful result. Consider the energy of a circular orbit r_o:

$$\begin{aligned}|E_c| &= \frac{1}{2}\frac{GmM}{r_o} \\ &= \frac{1}{2}mv^2 - \frac{GmM}{r}\end{aligned}$$

Rearranging for v, we arrive at the vis-viva equation:

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{2.17}$$

for a semi-major axis a. Note that despite deriving it by assuming circular motion, the result does hold true for any closed orbit. Now we are ready to tackle the problem at hand.

We can use the impulses Δv_1 and Δv_2, as shown in Figure (2.10), to achieve the desired result.

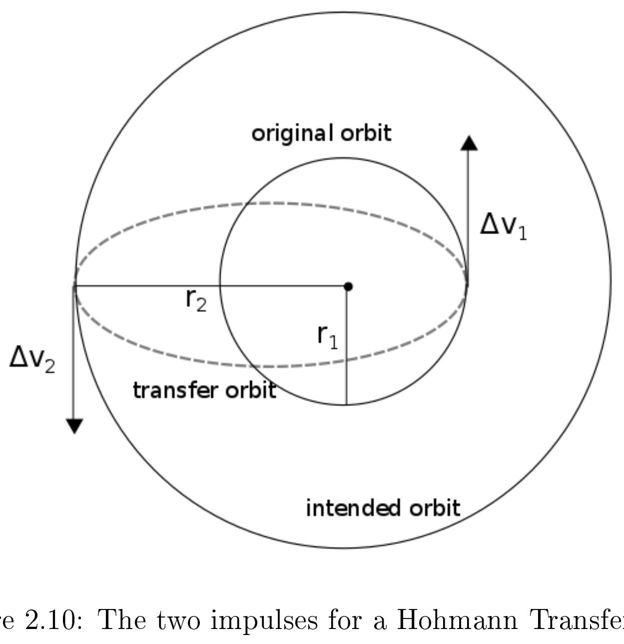

Figure 2.10: The two impulses for a Hohmann Transfer Orbit

Let us calculate each of the successive impulses. The initial orbital velocity is given by:

$$v_1 = \sqrt{\frac{GM}{r_1}}$$

At perigee, the elliptical transfer orbit has velocity:

$$\begin{aligned} v_1' &= \sqrt{GM\left(\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right)} \\ &= \sqrt{GM\left(\frac{2r_2}{r_1(r_1 + r_2)}\right)} \end{aligned}$$

Thus, the first impulse must have value:

$$\rightarrow \Delta v_1 = \sqrt{\frac{GM}{r_1}} \cdot \left(\sqrt{\frac{2r_1}{r_1 + r_2}} - 1\right)$$

The orbital velocity in the second circular orbit is given by:

$$v_2 = \sqrt{\frac{GM}{r_2}}$$

At apogee, the elliptical transfer orbit has velocity:

$$\begin{aligned} v_2' &= \sqrt{GM\left(\frac{2}{r_2} - \frac{2}{r_1 + r_2}\right)} \\ &= \sqrt{GM\left(\frac{2r_1}{r_2(r_1 + r_2)}\right)} \end{aligned}$$

Thus, the second impulse must have value:

$$\rightarrow \Delta v_2 = \sqrt{\frac{GM}{r_2}} \cdot \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

2.5 Lagrangian Mechanics

As stated in Section (1.1.1), Newton's Laws are a very cumbersome way to solve more complicated systems due to both the sheer computation required and the fact that it starts to becomes horrendously complicated to resolve forces. Lagrangian Mechanics allows us to solve such problems much more easily by considering the principle of stationary action; that the dynamics of a system are the minimization of a function that contains all the information in a system. Consider a function l:

$$l = \int_{x_o}^{x_1} f(y, y', x) \cdot dx$$

Writing the Taylor Expansion of the function f :

$$\begin{aligned}
\delta f &= \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + O(\delta y') \\
\delta y' &= \frac{d}{dx}(\delta y) \\
\delta l &= \int_{x_o}^{x_1} \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\left(\frac{\partial}{\partial x}\delta y\right) \cdot dx \\
&= \int_{x_o}^{x_1} \frac{\partial f}{\partial y}\delta y - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y \cdot dx
\end{aligned}$$

using integration by parts. We want $\delta l = 0$ for the functional to be minimised:

$$\begin{aligned}
0 &= \int_{x_o}^{x_1} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\delta y \cdot dx \\
\rightarrow \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= \frac{\partial f}{\partial y}
\end{aligned}$$

This principle of path minimisation is what the Lagrangian Method. In fact, we have just derived the functional form of the Euler-Lagrange equations, but more on this later. It is often quite helpful to default to using Lagrangian Mechanics if unsure of how to solve a problem, as this will often give way to a solution.

2.5.1 Definitions

First, we need to define a few key terms and concepts that are going to be referred to throughout the course of this section. They are as follows:

- Generalised Coordinates - A set of parameters q_k that specify the configuration of the system. These do not have to be orthogonal
- Degrees of Freedom - The number of independent coordinates that is sufficient to uniquely describe a system
- Holonomic Constraints - Constraints on a system are described as holonomic assuming that:
  1. The constraints are independent
  2. The system can be described by relations between general coordinate variables and time
  3. The number of generalised coordinates is reduced to the degrees of freedom of the system
- Noether's Theorem - Every differentiable symmetry of the action of a physical system has a corresponding conservation law

2.5.2 The Lagrangian

The Lagrangian ($\mathcal{L}$) is not unique to mechanics; in fact, it crops up in many different areas of physics. Generally, however, the Lagrangian is a function of the generalised coordinates (q_k), their time derivatives ($\dot{q_k}$) and time.

$$\mathcal{L} = \mathcal{L}(q_k,\ \dot{q_k},\ t)$$

In mechanics, the Lagrangian is simply defined as:

$$\mathcal{L} = KE - U \tag{2.18}$$

For example, we can define $\mathcal{L}$ some of the coordinate systems that we are familiar with:

- Cartesian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(x, y, z)$$

- Cylindrical:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - U(r, \theta, z)$$

- Spherical:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta \cdot \dot{\phi}^2\right) - U(r, \theta, \phi)$$

Now, *how do we use this?* The hardest part about using the Lagrangian is usually finding the Lagrangian itself; the rest is simply handle-turning. Once we have managed to find $\mathcal{L}$, we can find the equations of motion of the system using the *Euler-Lagrange Equations*:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_k}}\right) = \frac{\partial \mathcal{L}}{\partial q_k} \tag{2.19}$$

Note that for each coordinate and it's time derivative, we have one Euler-Lagrange (EL) equation. Note that the time derivative on the left-hand side is total; that is, which differentiate the coordinates contained in the expression with respect to time, rather than differentiate explicitly with respect to time. Don't worry if this seems a little confusing right now; have a read through some of the examples to becomes familiar with the process.

The EL equations actually have some interesting consequences due to Noether's Theorem (2.5.1). Essentially, if $\frac{\partial \mathcal{L}}{\partial q_k} = 0$, then the quantity $\frac{\partial \mathcal{L}}{\partial \dot{q_k}}$ is conserved. This is know as the *conjugate momentum*, denoted by p_k, and can be things such as linear or angular momentum that are constants of the motion.

2.5.3 The Hamiltonian

The Hamiltonian is a quantity derived from the Lagrangian, and is often useful for examining perturbations around some minimum of a system.

$$\mathcal{H} = \left(\sum_k \dot{q_k}\frac{\partial \mathcal{L}}{\partial \dot{q_k}}\right) - \mathcal{L} \tag{2.20}$$

Usually, $\mathcal{H}$ is written as a function of $\dot{p_k}$, $\dot{q_k}$ and time.

Like with the Lagrangian, the Hamiltonian has a set of equations that we use to derive the equations of motion, know as the *Hamiltonian Equations*:

$$\dot{p_k} = -\frac{\partial \mathcal{H}}{\partial q_k} \tag{2.21}$$

$$\dot{q_k} = \frac{\partial \mathcal{H}}{\partial p_k} \tag{2.22}$$

Note that the Hamiltonian itself is a characteristic of the motion of the system. Consider its total derivative:

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial p_k} \cdot \frac{\partial p_k}{\partial t} + \frac{\partial \mathcal{H}}{\partial q_k} \cdot \frac{\partial q_k}{\partial t} + \frac{d\mathcal{H}}{dt}$$

Using (2.20):

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{d\mathcal{H}}{dt}$$

This means that if $\mathcal{H}$ has no explicit time dependence, it is a conserved quantity for the system.

For small variations about a given minimum, we can either substitute $q_k = q_o + \delta q$ and expand (recalling that $\frac{\delta q}{q_o} \ll 1$) or we can use the concept of an effective potential:

$$U_{eff} = U_o + \frac{\partial U_o}{\partial q_k} \cdot q_k + \frac{\partial^2 U_o}{\partial {q_k}^2} \cdot \frac{{q_k}^2}{2!} + \cdots$$

$$\ddot{q_k} = -\frac{\partial^2 U_o}{\partial {q_k}^2} \cdot (q_k - q_o)$$

around a minimum q_o with a value of potential U_o at that point.

2.5.4 Example Problems

This material might seem complicated when reading over it (or it might not, who knows?), so the best way to become better acquainted with it is to read over and work through some examples.

1. A common problem is that of the compound pendulum; a non-uniform body that rotates about some axis that may not be through the centre of mass. *What is its period for small angle oscillations?*

   Let the pendulum have total mass m and moment of inertia I about the axis of rotation. Let us find the Lagrangian:

   $$U = -mgl\cos\theta$$
   $$KE = \frac{1}{2}I\dot{\theta}^2$$
   $$\rightarrow \mathcal{L} = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta$$

   Applying the EL equation:

   $$\frac{d}{dt}\left(I\dot{\theta}\right) = -mgl\sin\theta$$
   $$I\ddot{\theta} = -mgl\sin\theta$$

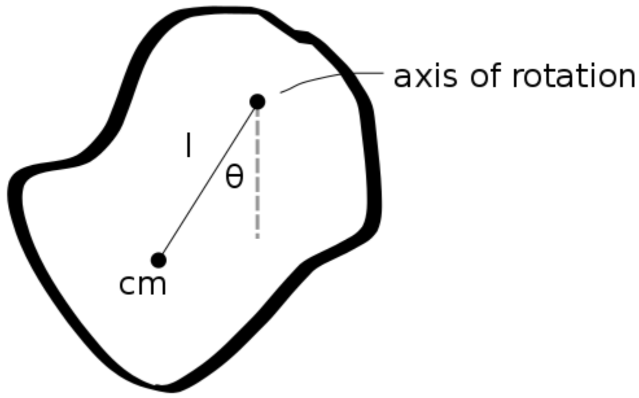

Figure 2.11: A typical compound pendulum

Using the small angle approximation that $\sin\theta \approx \theta$:

$$\ddot{\theta} + \frac{mgl}{I} \cdot \theta = 0$$

$$\rightarrow T = 2\pi\sqrt{\frac{I}{mgl}}$$

We can then substitute the moment of inertia of an appropriate body into this formula.

2. Suppose that two equal masses are joined by a string that runs over two pulleys. One of the masses is constrained to move in the vertical direction, while the other is free to swing freely, as shown in Figure (2.12). *Describe the evolution of the system for small angle oscillations of the right-hand mass of magnitude ϵ.*

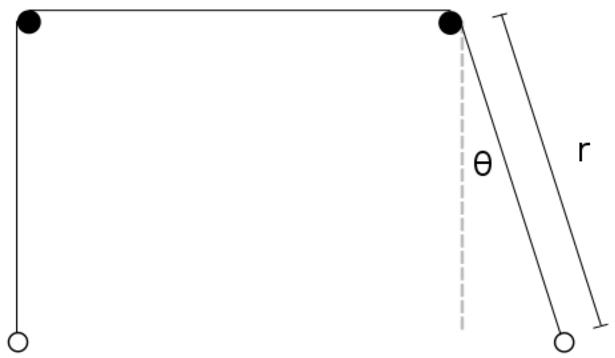

Figure 2.12: Two constrained masses in a pulley system

Let the distance from the wheel of the pulley to the right hand mass be r. Define the zero of potential of the masses as the equilibrium position when both masses are at rest and the system is balanced.

$$\begin{aligned} KE &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \\ U &= mgr - mgr\cos\theta \\ \rightarrow \mathcal{L} &= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr + mgr\cos\theta \end{aligned}$$

Applying the EL equations:

$$\begin{aligned} \frac{d}{dt}(2m\dot{r}) &= mr\dot{\theta}^2 - mg + mg\cos\theta \\ 2\ddot{r} &= r\dot{\theta}^2 - g(1-\cos\theta) \\ m\ddot{\theta} &= -mgr\sin\theta \\ \ddot{\theta} + \frac{g}{r}\sin\theta = 0 \end{aligned}$$

We are told that the amplitude of the oscillations is ϵ. Thus, the small angle solution to the θ coordinate equation is:

$$\begin{aligned} \theta(t) &= \epsilon\cos(\sqrt{\frac{g}{r}}t + \phi) \\ &= \epsilon\cos(\omega t + \phi) \end{aligned}$$

For small oscillations, $\cos\theta \approx 1 - \frac{\theta^2}{2}$. Substituting this into the r coordinate equation of motion:

$$\begin{aligned} 2\ddot{r} &= r\dot{\theta}^2 - g\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right) \\ &= r\dot{\theta}^2 - g\frac{\theta^2}{2} \\ &= r\left(\epsilon\omega\sin(\omega t + \phi)\right)^2 - \frac{g}{2}(\epsilon\cos(\omega t + \phi))^2 \\ &= \epsilon^2 g\left(\sin^2(\omega t + \phi) - \frac{1}{2}\cos^2(\omega t + \phi)\right) \end{aligned}$$

Averaging over a couple of periods:

$$\begin{aligned} 2\ddot{r} &= \frac{\epsilon^2 g}{4} \\ \ddot{r} &= \frac{\epsilon^2 g}{8} \end{aligned}$$

As $\epsilon > 0$, $g > 0$, this is a positive quantity, and so the left hand mass gradually climbs while the right hand mass gradually falls. This is a second order effect.

3. Consider a ball of mass m rolling without slipping on the surface of a hemisphere that is able to deform slightly. *Find the normal force. At what angle will the ball lose contact with the surface?*

As the surface is able to deform, we need to take account of the potential energy of this by a 'step' potential function $V(r)$.

$$\begin{aligned} KE &= \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2 \\ U &= mgr\cos\theta + V(r) \\ \rightarrow \mathcal{L} &= \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2 - mgr\cos\theta - V(r) \end{aligned}$$

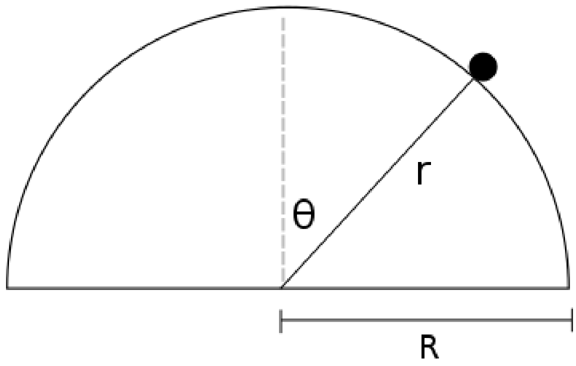

Figure 2.13: A ball rolling on a 'springy' hemisphere

Applying the EL equations, we obtain the equations of motion as:

$$mr^2\ddot{\theta} = mgr\sin\theta$$
$$m\ddot{r} = mr\dot{\theta}^2 - V'(r) - mg\cos\theta$$

We now place the constraints that $r = R$ and $\ddot{r} = \dot{r} = 0$.

$$0 = mR\dot{\theta}^2 - \frac{dV}{dr}\bigg|_{r=R} - mg\cos\theta$$
$$-\frac{dV}{dr}\bigg|_{r=R} = mg\cos\theta - mR\dot{\theta}^2$$

However, recall (1.3). The reaction force is thus:

$$F_r(\theta, \dot{\theta}) = mg\cos\theta - mR\dot{\theta}^2$$

The ball will lose contact with the surface the instant that $F_r = 0$.

$$0 = mg\cos\theta - mr\dot{\theta}^2$$
$$mg\cos\theta = mR\dot{\theta}^2$$
$$\cos\theta = \frac{R}{g}\left(\frac{v}{R}\right)^2$$
$$= \frac{v^2}{Rg}$$

By the conservation of energy from when the ball is initially at rest at the top of the hemisphere:

$$\frac{1}{2}mv^2 = mgR(1 - \cos\theta)$$
$$v^2 = 2gR(1 - \cos\theta)$$
$$R\cos\theta = 2R(1 - \cos\theta)$$
$$3R\cos\theta = 2R$$
$$\cos\theta = \frac{2}{3}$$

Hence, the particle will leave the surface at $\theta_{max} = \cos^{-1}\left(\frac{2}{3}\right)$.

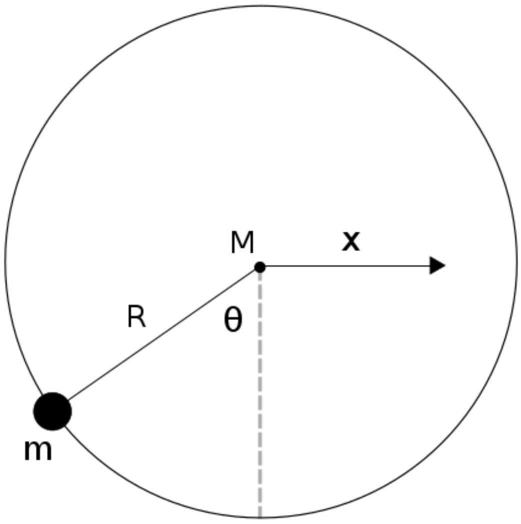

Figure 2.14: A point mass attached to the rim of a disk

4. Imagine that a point mass of mass m is attached to the rim of a disk of mass M. *What is the value of the period for small angle oscillations amount the minimum of the motion?*

Let the zero of potential be at the centre of the wheel. Now, assuming that the wheel rolls without slipping, the position x of the centre of the wheel must satisfy the condition $x = R\theta$. We can thus write the position of the smaller mass as:

$$\begin{aligned} \underline{r} &= (x - R\sin\theta,\ R(1-\cos\theta)) \\ &= (R(\theta\sin\theta),\ R(1-\cos\theta)) \\ \dot{r} &= R(\dot\theta - \dot\theta\cos\theta,\ \dot\theta\sin\theta) \end{aligned}$$

Considering $\dot{r}^2$ to find the kinetic energy:

$$\begin{aligned} \dot{r}^2 &= R^2(\dot\theta^2 - 2\dot\theta^2\cos\theta + \dot\theta\cos^2\theta + \dot\theta^2\sin^2\theta) \\ &= 2R^2\dot\theta^2(1-\cos\theta) \\ KE &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(R^2\dot\theta^2)(1-\cos\theta) \\ &= \frac{1}{2}MR^2\dot\theta^2 + \frac{1}{2}m(R^2\dot\theta^2)(1-\cos\theta) \\ U &= -mgR\cos\theta \\ \rightarrow \mathcal{L} &= \frac{1}{2}MR^2\dot\theta^2 + \frac{1}{2}m(R^2\dot\theta^2)(1-\cos\theta) + mgR\cos\theta \end{aligned}$$

We have thus derived a Lagrangian for the system that only has one degree of freedom. Applying the EL equations:

$$\begin{aligned} \frac{d}{dt}\Big(MR^2\dot\theta + 2mR^2\dot\theta(1-\cos\theta\Big) &= -\frac{1}{2}(2mR^2\dot\theta^2)\sin\theta - mgR\sin\theta \\ MR^2\ddot\theta + 2mR^2\ddot\theta(1-\cos\theta) &= -mR^2\dot\theta^2\sin\theta - mgR\sin\theta \\ \ddot\theta\left(MR^2 + 2mR^2(1-\cos\theta)\right) &= -\sin\theta(mR^2\dot\theta^2 + mgR) \end{aligned}$$

For small oscillations, let $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$. Neglecting terms of $O(\theta^2)$

and above:

$$\ddot{\theta}(MR^2) + mgR\sin(\theta) = 0$$
$$\ddot{\theta} + \frac{mg}{MR}\theta = 0$$

Hence the frequency of small angle oscillations about equilibrium is:

$$\Omega = \sqrt{\frac{mg}{MR}}$$

5. A bead of mass m slides around inside a cone of semi-vertical angle α. The bead is constrained to move on the surface of the cone. *Find the frequency of small oscillations about a stable equilibrium, and find the value for α for which this frequency is equal to the frequency at stable equilibrium.*

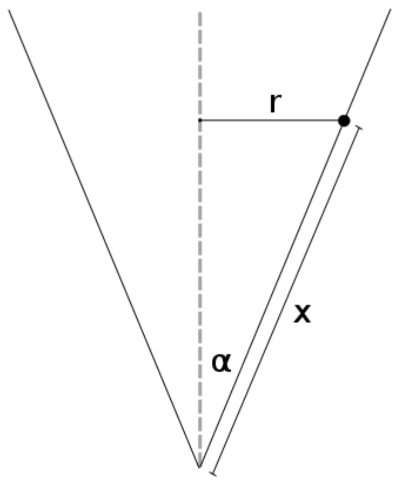

Figure 2.15: A bead sliding around inside a cone

Take the zero of potential to be at the vertex of the cone.

$$\tan\alpha = \frac{r}{h}$$
$$h = \frac{r}{\tan\alpha}$$
$$U = mgh$$
$$= \frac{mgr}{\tan\alpha}$$

Similarly,

$$\sin\alpha = \frac{r}{x}$$
$$x = \frac{r}{\sin\alpha}$$
$$\dot{x} = \frac{\dot{r}}{\sin\alpha}$$
$$KE = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right)$$

Hence we can write the Lagrangian of the system as:

$$\rightarrow \mathcal{L} = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - \frac{mgr}{\tan\alpha}$$

Now, applying the EL equations:

$$\begin{aligned}
\frac{m\ddot{r}}{\sin^2\alpha} &= -\frac{mg}{\tan\alpha} + mr\dot{\theta}^2 \\
\ddot{r} &= -\frac{mg}{\sin\alpha}\cos\alpha\sin^2\alpha + r\dot{\theta}^2\sin^2\alpha \\
&= r\dot{\theta}^2\sin^2\alpha - g\cos\alpha\sin\alpha
\end{aligned}$$

Evidently, $p_\theta = L = mr^2\dot{\theta}$ is a conserved quantity; this is the angular momentum of the bead around the $\hat{\underline{z}}$ axis. Let us re-write the equation of motion in terms of this quantity:

$$\begin{aligned}
\dot{\theta} &= \frac{L}{mr^2} \\
\ddot{r} &= \frac{L^2\sin^2\alpha}{m^2r^3} - g\cos\alpha\sin\alpha
\end{aligned}$$

We have reduced this to a one dimensional problem in the r coordinate.

To find the frequency of stable circular motion, we need to impose the conditions that $\ddot{r} = 0$, $\dot{r} = 0$, $r = r_o$.

$$\begin{aligned}
0 &= r_o\dot{\theta}^2\sin^2\alpha - g\cos\alpha\sin(\alpha) \\
\dot{\theta}^2 r_o\sin^2\alpha &= g\cos\alpha\sin\alpha \\
\rightarrow \dot{\theta}_c &= \sqrt{\frac{g}{r_o\tan\alpha}}
\end{aligned}$$

Now for small oscillations about equilibrium, we let $r = r_o + \delta$.

$$\begin{aligned}
\ddot{\delta} &= \frac{L^2\sin^2\alpha}{m^2(r_o+\delta)^3} - g\cos\alpha\sin\alpha \\
&= \frac{L^2\sin^2\alpha}{m^2{r_o}^3\left(1+\frac{\delta}{r_o}\right)^3} - g\cos\alpha\sin\alpha
\end{aligned}$$

Binomially expanding using the fact that $r_o \gg \delta$:

$$\begin{aligned}
\ddot{\delta} &= \frac{L^2\sin^2\alpha}{m^2{r_o}^3}\left(1 - 3\frac{\delta}{r_o}\right) - g\cos\alpha)\sin\alpha \\
&= \frac{L^2\sin^2\alpha}{m^2{r_o}^3} - 3\frac{L^2\sin^2\alpha}{m{r_o}^4}\delta - g\cos\alpha\sin\alpha \\
&= -3\frac{L^2\sin^2\alpha}{m{r_o}^4}\delta
\end{aligned}$$

Substituing in for L, we obtain:

$$\ddot{\delta} + \frac{3g}{r_o}\sin\alpha\cos\alpha\cdot\delta = 0$$

$$\rightarrow \Omega = \sqrt{\frac{3g}{r_o}\sin\alpha\cos\alpha}$$

For this to equal the frequency of circular motion, we require:

$$\Omega = \dot{\theta}_c$$

$$\frac{3g}{r_o}\sin\alpha\cos\alpha = \frac{g}{r_o\tan\alpha}$$

$$sin^2\alpha = \frac{1}{3}$$

$$\rightarrow \alpha = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

3. *Special Relativity*

This chapter aims to cover the basics of Special Relativity, including:

- The Michaelson-Morley Experiment and Einstein's Postulates
- The Lorentz Transformations
- Space-Time Diagrams
- Relativistic Dynamics
- The Relativistic Doppler Effect

This turns out to be one of the more interesting topics of the mechanics course, as it has consequences that not many students will have encountered or considered before. However, some of the concepts can be difficult to grasp, so it is worth working through problems fully to get to grips with the material. Note that for the entirety of this chapter, the symbol γ is given by:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

where c is the speed of light in a vacuum, and v is the velocity of the moving body. Sometimes, we use $\beta = \frac{v}{c}$ to simplify some expressions.

3.1 The Michaelson-Morley Experiment and Einstein's Postulates

In special relativity, an *event* is defined as a point in both space and time (more commonly referred to as 'space-time') that has an associated physical occurrence. Unlike what we currently think of an event as, such as "meeting up with friends for coffee" or "going to the cinema", events in special relativity are used to refer to a specific instance in time, such as the emission of light instantaneously from a source, or its reception at another point in space time. An event can be described by a time coordinate, and a set of spatial coordinates. For example, we can define an event E by $E\,(ct,\, x,\, y,\, z)$. This is using what is known as a *four-vector*. It consists of a time component, followed by three spatial components. Here it is being used to represent the space-time coordinates of an event, but it can also be used to represent the energy and momentum of a particle, amongst other things.

A *frame* is a convenient standard of rest or coordinate system in which we have chosen to make our measurements. An inertial reference frame is one in which (1.1) is obeyed as it is not accelerating. For example, imagine a car accelerating away from a set of traffic lights that just turned green. We can observe the scenario from the inertial frame of a stationary person standing on the pavement, or in the non-inertial frame of the accelerating car. Note that the term 'stationary' is used in this case in a relative sense; we have defined an object to be stationary if it has no velocity in the frame of the pavement, but this is not a universal standard of rest.

3.1.1 The Aether and The Michaelson-Morley Experiment

Now, the question arises: *what frame of reference do we measure light relative to?* One such theory was that it propagated through a medium called the 'aether' at the speed of light. The aether was all permeating and had zero viscosity (properties that these days would quickly ring alarm bells in the head of any switched on physicist!). The speed of light would thus be measured based on the relative motion between a body and the aether.

The Michaelson-Morley experiment was designed to test this theory, using the apparatus shown in Figure (3.1). We assume that the apparatus is moving at a speed v with respect to the aether.

Consider the time taken for the light to travel along each of the perpendicular paths. First, t_1 along the horizontal path:

$$\begin{aligned} t_1 &= \frac{l}{c+v} + \frac{l}{c-v} \\ &= \frac{2l}{c\left(1-\frac{v^2}{c^2}\right)} \\ &\approx \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right) \end{aligned}$$

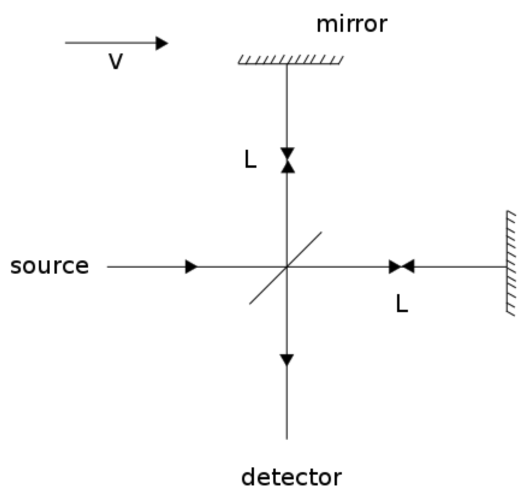

Figure 3.1: A schematic drawing of the setup of the M-M experiment

Then, t_2 along the vertical path:

$$\begin{aligned} t_2 &= \frac{2l}{\sqrt{c^2 - v^2}} \\ &= \frac{2l}{c\sqrt{1 - \frac{v^2}{c^2}}} \\ &\approx \frac{2l}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \end{aligned}$$

Finally, taking the time difference:

$$\begin{aligned} \Delta t &= t_1 - t_2 \\ &= \frac{lv^2}{c^3} \\ &\neq 0 \end{aligned}$$

The M-M experiment thus predicted that there would be a time difference in the arrival of light at the detector that would cause a shift in the background interference pattern present. However, no such shift was observed. This null result contributed to the destruction of the aether model, and it's eventual replacement by Einstein's Postulates.

3.1.2 Einstein's Postulates

Einstein's paper on the Special Theory of Relativity rests on two main postulates:

1. The outcome of any experiment is the same in any inertial frame of rest
2. Light in a vacuum travels at a constant speed c at all times, regardless of one's frame of observation

The first is essentially a statement of the idea that no fictitious forces are apparent in inertial frames. The second is really where things start to complicate themselves; this simple statement that light is constant for all observers has quite remarkable consequences. Note that a secondary consequence of this postulate is that no information can be transmitted above the speed of light; it is the universal ceiling (if you like) for information transmission.

One such consequence (more to follow in later sections) is that events that appear simultaneous in one frame may not be so in another. Consider the 'thought experiment' of two observers; one located equidistant from each ends of a train moving at a relativistic speed v, and the other on the stationary platform.

Figure 3.2: A diagram showing the lengths involved in the Train thought experiment

Lights at either end of the train (A and B) are turned on. First, consider the time taken for the light to travel along OA (t_1) and OB (t_2) i.e in the rest frame of the observer inside the train.

$$t_1 = \frac{OA}{c} = \frac{L}{c}$$
$$t_1 = t_2$$

This means that for the observer on the train observes the lights at A and B to turn on simultaneously. *What about the observer on the platform?*

$$t_1{}' = \frac{OA'}{c} = \frac{L - v\left(\frac{L}{c}\right)}{c}$$
$$t_2{}' = \frac{OB'}{c} = \frac{L + v\left(\frac{L}{c}\right)}{c}$$
$$t_1 \neq t_2$$

This means that the observer on the platform observes the lights turning on at different times (B before A). This, it is clear that whether or not two events are simultaneous will depend on the frame of reference from which they are being observed.

3.2 The Lorentz Transformations

At this stage, it is useful to define two key terms:

- Proper Time - The shortest time interval between two events, as measured in the frame at which the relevant clock is at rest
- Proper Length - The length of a body as observed in its own reference frame

As we have seen, our perception of length and time can change depending on the frame that we find ourselves in. We can relate these quantities in two frames by *The Lorentz Transformations* (LT's). An event in frame S defined by (ct, x, y, z) is related to an event (ct', x', y', z') in frame S' by:

$$x' = \gamma(x - vt) \tag{3.1}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{3.2}$$

We can also consider the transformation of energy E and momentum p between these frames:

$$E' = \gamma(E - vp) \tag{3.3}$$

$$p' = \gamma\left(p - \frac{vE}{c^2}\right) \tag{3.4}$$

These can also be represented as a matrix, which is useful if we want to perform successive LT's. It also implies that the Lorentz transformations are in some sense a 'rotation' in space-time (remark how the matrix is symmetric).

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.5}$$

We have included the zero entries to impress upon the reader the importance of the fact that the LT's apply to all directions, even though we are dealing with only the one-dimensional case at this stage.

Note that we re-obtain the conventional Galilean Transformations for $v \ll c$.

$$\gamma \to 1$$
$$t' \to t$$
$$x' \to x - vt$$

This to be expected, as the effects of special relativity are negligible at low velocities.

3.2.1 The Interval

The LT's have a particular property in that they preserve a quantity that we call 'the interval'. The interval s of a particular set of space-time coordinates is defined as:

$$s^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{3.6}$$

This is analogous to the dot product of a four-vector with itself. A true rotational matrix leaves the length of a vector completely unchanged, but in the LT case it is only the interval that is an *invariant quantity*; a quantity that does not change regardless of one's frame of reference.

Two observers may find different space and time separations of two events, *but they will both agree on the interval that they calcuate.* The use of such invariant quantities becomes more relevant later on when we start tackling relativistic dynamics, but we have introduced it here so the reader becomes more familiar with it reading through the subsequent sections.

3.2.2 Time Dilation

One consequence of Einstein's Postulates is the effect of time dilation; *that time appears to pass slower on objects moving relative to the observer.* When first encountered, this appears as a bit of an odd effect, as it holds for whatever frame you place yourself in. For example, let observer A be in a relativistic rocket moving past an observer B on the Earth. From B's frame of reference, the time appears to be passing more slowly on the clock in A's frame of reference because B observes A to be moving relative to their apparently stationary frame on the earth. Conversely, A observes time to be passing more slowly on the clock in B's frame of reference, as from this frame *the earth appears to be moving, while the rocket remains stationary* relative to A.

Consider the proper-time interval $\Delta t' = t_2' - t_1'$ between two events as measured by a clock that is stationary in frame S', which itself is moving at a velocity v relative to frame S. Using (3.2), the time interval between these two events as perceived in frame S is given by:

$$\begin{aligned}\Delta t &= t_2 - t_1 \\ &= \gamma\left(t_2' - \frac{vx_2'}{c^2}\right) - \gamma\left(t_1' - \frac{vx_1'}{c^2}\right) \\ &= \gamma(t_2' - t_1') - \gamma\frac{v}{c^2}(x_2' - x_1')\end{aligned}$$

However, these two events are measured by a clock that is stationary in frame S'. This means that $x_2' = x_1'$. Hence, we find that:

$$\Delta t = \gamma \Delta t' \tag{3.7}$$

Remember when trying to apply this equation that *time is dilated on objects moving relative to your frame of reference.*

3.2.3 Length Contraction

Similar to time dilation, length contraction describes how *lengths in frames moving relative to the observer appear contracted.* Note that this contraction only occurs in the direction of motion. For example, if a spacecraft is rushing past the Earth at a relativistic speed, it will appear 'squashed' in it's direction of motion, but have the same dimension perpendicular to it.

Consider the proper length interval $\Delta x = x_2 - x_1$ at rest in frame S. Using (3.2), the length interval between these two events as perceived in frame S' is given by:

$$\begin{aligned}\Delta x &= x_2 - x_1 \\ &= \gamma(x_2' - vt_2') - \gamma(x_1' - vt_1') \\ &= \gamma(x_2' - x_1') - \gamma(t_2' - t_1')\end{aligned}$$

However, these measurements are taken at the same time in frame S', meaning that $t'_2 = t'_1$. Hence we find that:

$$\Delta x' = \frac{\Delta x}{\gamma} \tag{3.8}$$

Remember that when trying to apply this equation that *length is contracted on objects moving relative to your frame of reference.*

3.2.4 Relativistic Velocity Addition

Imagine that we have two spacecraft travelling along an axis towards each other. Spacecraft U has velocity $u = 0.8c$ and Spacecraft V has velocity $v = 0.9c$. If we attempt to find their relative velocity by using the 'classical' method of adding their velocities, we obtain $u'_{rel} = 1.7c$. However, we know that as a consequence of Einstein's Postulates, nothing can travel faster than the speed of light.

In Special Relativity, u'_{rel} is known as the mutual velocity of A and B; this does not actually violate the limitation of the speed of light as no information travels at this velocity. If we wanted to beam a signal between the two ships and measure their relative velocity, we would not obtain v_{rel}. So the question becomes: *how do we add velocities in relativity?*

For this, we want to consider these spacecraft U and V moving *away* from one another along the x-axis with velocities u and v respectively. To find the mutual velocity of the two objects, we want to transform the rate of change of the distance between the two objects in the rest frame of a stationary observer into the frame of one of the two spacecraft. For simplicity's sake, we are going to perform this derivation assuming only one-dimensional motion, but it can be generalised to three dimensions. Let us transform into the frame of spacecraft U. Considering (3.2) in their differential form:

$$\begin{aligned} u_{rel} &= \frac{dx'}{dt'} \\ &= \frac{\gamma(dx - vdt)}{\gamma\left(dt - v\frac{dx}{c^2}\right)} \\ &= \frac{dx - vdt}{dt - v\frac{dx}{c^2}} \\ &= \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} \end{aligned}$$

But $\frac{dx}{dt} = u$. Thus, we obtain a formula for relativistic velocity addition of:

$$\underline{u}_{rel} = \frac{\underline{u} - \underline{v}}{1 - \frac{\underline{u} \cdot \underline{v}}{c^2}} \tag{3.9}$$

We have gone back to vector notation here to remind the reader that $\underline{u}$ and $\underline{v}$ are in fact vector quantities.

Let us go back to our example of the spacecraft A and B moving towards one another. As before, we found that the mutual velocity was $1.7c$. Now using (3.9):

$$u_{rel} = \frac{v_A + v_B}{1 + \frac{v_A v_B}{c^2}}$$

Note the signs have changed as the objects are moving *towards* one another in this problem.

$$\begin{aligned} u_{rel} &= \frac{0.8c + 0.9c}{1 + (0.8)(0.9)} \\ &\approx 0.988c \end{aligned}$$

Now, *what if A and B were moving away from one another?*

$$\begin{aligned} u_{rel} &= \frac{v_A - v_B}{1 - \frac{v_A v_B}{c^2}} \\ &= \frac{0.8c - 0.9c}{1 - (0.8)(0.9)} \\ &\approx -0.357c \end{aligned}$$

As we have shown, (3.9) gives very different results to simply adding the velocities; this is the nature of Special Relativity!

3.3 Space-Time Diagrams

Space-time diagrams are useful tools for analysing problems of Special Relativity. In essence, they are a 'graph' of events in space time, with spatial coordinates on the x-axis, and time coordinates on the y-axis. Each event in time is represented by a *world line*, and has an associated *light cone*.

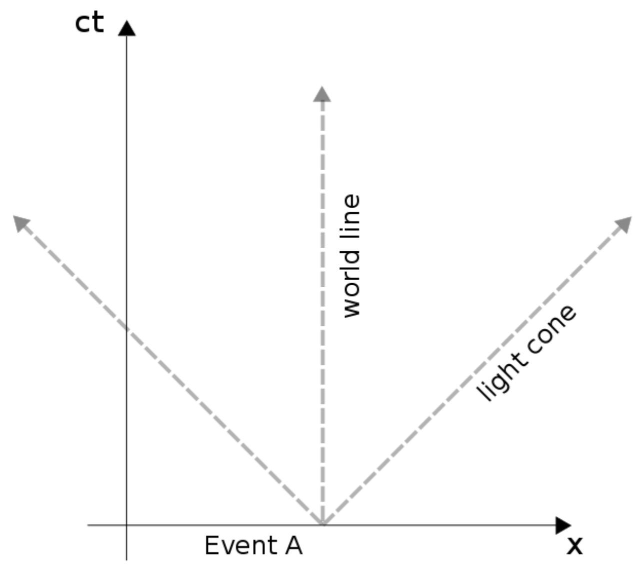

Figure 3.3: An example space-time diagram

Event A can only effect events that lie inside it's light cone. Intervals inside the light cone obey $(\Delta s)^2 > 0$, and are referred to as 'time-like'. On the light cone itself, $(\Delta s)^2 = 0$. This is said to be 'light-like'. Elsewhere, $(\Delta s)^2 < 0$, and this is said to be 'space-like'.

Objects or events that remain fixed in space-time have vertical world lines, but objects or events that move in space-time have slanted world lines. The faster the speed, the less steep the world line. By convention, world lines that are at 45° represent objects or events travelling at the speed of light. This means that no world line can be at less than 45°.

3.3.1 An Example

A stationary observer A fires a light pulse at a mirror located at a distance d away. A second observer B is moving at a relativistic speed v along the line of projection of the light pulse. *What is the time interval between the transmission and reception of the signal by A from B's frame of reference?*

We can solve this problem quite effectively by the use of space-time diagrams in both A's and B's rest frame.

Let Event 1 be the point at which the light hits the mirror, and Event 2 the time the light returns to observer A. For Event 1:

$$x_1 = d$$
$$ct_1 = ct_o$$

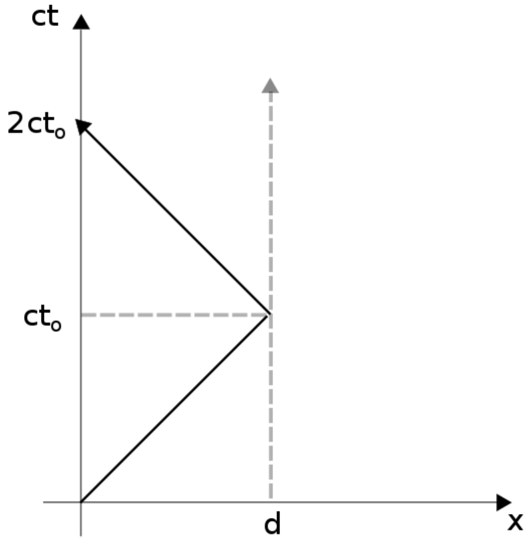

Figure 3.4: A space-time diagram in the rest frame of A

Applying the LT's:

$$x_1' = \gamma(d - c\beta t_o)$$

For Event 2:

$$x_2 = 0$$
$$ct_2 = 2ct_o$$
$$x_2' = -2\gamma\beta ct_o$$

This means that the space-time diagram in the rest frame of B becomes:

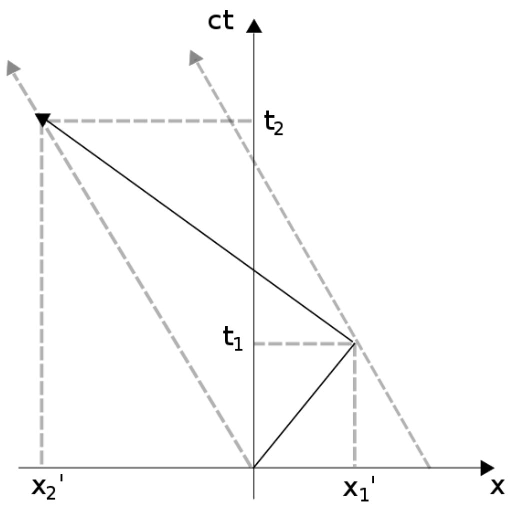

Figure 3.5: A space-time diagram in the rest frame of B

This is because the world lines of the observer A and the mirror at a distance d appear to be moving in the rest frame of B. Now the time interval in B's frame of reference is then given by:

$$\begin{aligned}
t_B &= t_1 + t_2 \\
&= \frac{\gamma}{c}(d - c\beta t_o) + \frac{\gamma}{c} \cdot 2\beta t_o \\
&= \gamma \left(\frac{d}{c} + \beta t_o \right) \\
&= \gamma \left(\frac{d}{c} + \beta \frac{d}{c} \right) \\
&= \frac{d}{c} \cdot \frac{\sqrt{(1+\beta)^2}}{\sqrt{(1-\beta)(1+\beta)}} \\
\rightarrow t_B &= \frac{d}{c} \cdot \sqrt{\frac{1+\beta}{1-\beta}}
\end{aligned}$$

As you can see, the calculation is made easier to visualise and understand what is going on when we use both the LT's and space-time diagrams in conjunction with one another.

3.4 Relativistic Dynamics

Unlike with Classical Dynamics, we cannot simply say that if a body is at rest at zero potential it has zero energy; we have not taken account of the *rest mass energy* of the body. In Special Relativity, the energy and momentum of a body are given by:

$$E = \gamma mc^2 \tag{3.10}$$
$$\underline{p} = \gamma m\underline{v} \tag{3.11}$$
$$KE = (\gamma - 1)mc^2 \tag{3.12}$$

Consider the following:

$$E^2 - p^2c^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 \tag{3.13}$$
$$= \gamma^2 m^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \tag{3.14}$$
$$= m^2c^4 \tag{3.15}$$

This is an invariant quantity because the rest mass m of a body can never change, making it an important (and useful) result. We can show that it is explicitly frame invariant by considering (3.4):

$$\begin{aligned} E'^2 - p'^2c^2 &= \gamma^2(E - vp)^2 - \gamma^2\left(pc - \frac{vE}{c}\right)^2 \\ &= \gamma^2\left(E^2 - p^2c^2 - \left(\frac{v^2E^2}{c^2} + v^2p^2\right)\right) \\ &= \gamma^2\left(E^2 - p^2c^2\right)\left(1 - \frac{v^2}{c^2}\right) \\ &= E^2 - p^2c^2 \\ &= m^2c^4 \end{aligned}$$

We can actually represent the momentum and energy of a body by *four-vector momentum.* Four vector is of the form:

$$\underline{p}^{\mu} = \left(\frac{E}{c}, p, 0, 0\right)$$

Again, we have left the zeroes in to demonstrate that the momentum can be in any direction, and not just along the x-axis. As it contains the information for all the energy and momentum of a system, *four-vector momentum is conserved.* This means that we can equate four-vector momenta if the momenta are written in the same frame. Like the interval s, the dot product of a four-vector with itself yields an invariant quantity:

$$\begin{aligned} \underline{p}^{\mu} \cdot \underline{p}^{\mu} &= \frac{E^2}{c^2} - p^2 \\ &= m^2c^2 \end{aligned}$$

This means that we can equate *the dot product of the sum of four-vector momenta*, even if the momenta are written in different frames. Do not worry if this seems a little confusing right now; there are plenty of examples to follow that will help elucidate these concepts.

Like with the LT's, these results reduce to the Classical results for $v \ll c$. For example, consider the kinetic energy:

$$\begin{aligned} KE &= (\gamma - 1)mc^2 \\ &\approx \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right)mc^2 \\ &= \frac{1}{2}mv^2 \end{aligned}$$

Some of you might be wondering at this point about *how fast we need to be going for these relativistic effects to matter?* Well, if we again consider the kinetic energy:

$$\begin{aligned} KE &= (\gamma - 1)mc^2 \\ &\approx \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^2}{c^4} - 1\right)mc^2 \\ &= \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} \end{aligned}$$

For us to have a 1% error in kinetic energy, we require:

$$\begin{aligned} 0.01 &= \frac{\frac{3}{8}\beta^4}{\frac{1}{2}\beta^2} \\ &= \frac{3}{4}\beta^2 \\ \beta &\approx 0.115 \end{aligned}$$

Thus, a particle has to be going at just over 10% of the speed of light for relativistic effects to become noticeable.

3.4.1 Photons

So what happens when the particle that we are considering is a photon? By definition, a photon has zero rest mass energy. This means that *the square of the four-momentum for any photon is always zero.* Furthermore, this result leads to:

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{3.16}$$

where h is Plank's Constant. This means that a photon still has momentum even though it's rest mass is zero. This does not make sense classically, but if considered relativistically:

$$\begin{aligned} \lim_{v \to c} p &= \lim_{v \to c} \gamma m v \\ &= (\infty)(o)c \end{aligned}$$

which has a finite value.

3.4.2 Approximations

In relativity, there are a number of useful approximations that we can make to simplify calculations.

- If a particle is very light, and $E \gg mc^2$, then by (3.15):

$$E \approx pc$$

- For a particle travelling very quickly such that $v \approx c$, then:

$$\sqrt{\frac{1-\beta}{1+\beta}} \approx \sqrt{\frac{2}{1-\beta}}$$
$$\frac{v}{\sqrt{1-\beta^2}} \approx \frac{c}{\sqrt{2(1-\beta)}}$$

The wording of questions will often make it obvious when these are to be used. If you get stuck on a question, look to see if any approximations such as these can be made.

3.4.3 Example Problems

Below are a range of problems covering different aspects of the material in this section. They should serve as examples that can be referenced when working on harder or more involved problems.

1. Let us examine the following question: *can an isolated electron emit a photon?*

   Let the original electron have energy E_e, the photon energy E_γ and the energy of the final electron be E_e'.

$$E_e{}' = \sqrt{m_e{}^2c^4 + p_e{}^2c^2}$$
$$E_\gamma = p_\gamma c$$

   Now considering the conservation of energy:

$$E_\gamma + E_e{}' = E_e$$
$$p_\gamma c + \sqrt{m_e{}^2c^4 + p_e{}^2c^2} = m_e c^2$$

   By conservation of momentum, $p_e = p_\gamma$. Hence we obtain:

$$E_\gamma + \sqrt{m_e{}^2c^4 + E_\gamma{}^2} = m_e c^2$$

   This relationship can only hold if $E_\gamma = 0$, meaning that an isolated electron *cannot* emit a photon.

2. In a similar way to the previous problem, we will pose the question: *Can the collision of a positron and electron create a single photon?*

   Let us first consider the four-vector momenta:

$$\underline{p}_1^\mu = \left(\frac{E_1}{c}, p_1, 0, 0\right)$$
$$\underline{p}_2^\mu = \left(\frac{E_2}{c}, -p_2, 0, 0\right)$$
$$\underline{p}_3^\mu = \left(\frac{E_3}{c}, \frac{E_3}{c}, 0, 0\right)$$

   By the conservation of four-vector momentum:

$$\underline{p}_1^\mu + \underline{p}_2^\mu = \underline{p}_3^\mu$$
$$\underline{p}_1^\mu = \underline{p}_3^\mu - \underline{p}_2^\mu$$

Squaring both sides to obtain the invariant quantities:

$$m^2c^2 = m^2c^2 + 2\frac{E_3}{c}p_2 - 2\frac{E_2E_3}{c}$$
$$\rightarrow p_2 = \frac{E_2}{c}$$

However, as an electron is a massive particle - that is, it has rest mass energy - the final relation cannot hold. This means that we have shown that the proposed process is *not* possible.

3. An electron of energy 9.0 GeV and a positron of energy E collide head-on to produce a B^0 meson and an anti-B^0 meson, each with a mass of 5.3 Gev/c^2. *What is the threshold energy required to produce the B^0 meson pair?*

Using four vector invariance across the laboratory (left hand side) and the centre of mass (right hand side) frames:

$$(9+E)^2 - (\underline{p}c)^2 = (2m_Bc^2)^2$$

In this case, $E \gg m_ec^2$. This means that we can use the first approximation in Section (3.4.2). Thus the net momentum is given by:

$$p = \frac{9}{c} - \frac{E}{c}$$

Substituting this result in and re-arranging:

$$(9+E)^2 - (9-E)^2 = (2m_Bc^2)^2$$
$$36E = 4(5.3)^2$$
$$\rightarrow E \approx 3.12\,GeV$$

The B^0 mesons undergo decay with a mean proper life-time of $1.5 \times 10^{-12}\,s$. Assuming that the particles are produced at threshold energy, *what is the mean distance that the B^0 mesons travel in the lab frame before they decay?*

Consider the conservation of energy in the lab frame:

$$9 + E = 2\gamma m_Bc^2$$

We can substitute our result for E back in to find the γ factor for the B^0 mesons. This gives $\gamma \approx 1.14$. From the lab. frame, the life-time of the particles appears much longer; we thus have to consider the dilated life-time of the particles. Using (3.7):

$$\tau' = \gamma\tau$$
$$= (1.5 \times 10^{-12})(1.14)$$
$$= 1.71 \times 10^{-12}\,s$$

Finding the velocity:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$
$$\approx 0.485c$$

Thus the mean distance they travel is given by:

$$\begin{aligned} < r > &= (0.485 \times 3 \times 10^8)(1.71 \times 10^{-12}) \\ &= 2.5 \times 10^{-4}\, m \end{aligned}$$

We can ask one further question (last one!): *What is the probability that one of the photons decays before reaching this mean distance?*

As the particles decay, they must obey the population decay equation:

$$\frac{dN}{dt} = -\lambda N$$

This has solution:

$$\frac{N}{N_o} = e^{-\frac{t}{\tau}}$$

where N_o is the initial population. The associated that a particle survives/decays over a time t_o are:

$$P(Survive) = e^{-\frac{t_o}{\tau}} \tag{3.17}$$

$$P(Decay) = 1 - e^{-\frac{t_o}{\tau}} \tag{3.18}$$

Using $t_o = 1.71 \times 10^{-12}$ (time to travel the mean distance) and $\tau = 1.71 \times 10^{-12}$ (the mean life-time), we obtain:

$$P(Decay) \approx 63\%$$

This example covers a lot of the basic concepts involved in relativistic collisions that produce decaying particles.

4. A photon of unknown energy E collides head-on with a photon of energy ε for $\varepsilon \ll E$. *What is the minimum value of E required to produce an electron-positron pair?*

Again, let us consider the four-vector momenta of the constituents.

$$\begin{aligned} \underline{p}_1^\mu &= \left(\frac{E}{c}, \frac{E}{c}, 0, 0\right) \\ \underline{p}_2^\mu &= \left(\frac{\varepsilon}{c}, -\frac{\varepsilon}{c}, 0, 0\right) \end{aligned}$$

We have left out the four-vector momenta for the electrons; we don't have to worry about this as we can just consider them to be created in the zero-momentum (ZMF)/centre-of-mass frame, and equate invariants as follows:

$$\begin{aligned} \left(\underline{p}_1^\mu + \underline{p}_2^\mu\right)^2 &= \left(\underline{p}_e^\mu + \underline{p}_e^\mu\right)^2 \\ 4{m_e}^2 c^2 &= \frac{4E\varepsilon}{c^2} \\ \rightarrow E &= \frac{{m_e}^2 c^4}{\varepsilon} \end{aligned}$$

What is the speed of the ZMF in this case?

As the energy of the second photon is so small (ε), we can neglect its energy when transforming to the ZMF. This means we can easily calculate the energy in the ZMF, and thus find it's speed:

$$E_{ZMF} = (E+\varepsilon)\sqrt{\frac{1-\beta}{1+\beta}}$$
$$2m_ec^2 \approx E\sqrt{\frac{1-\beta}{2}}$$
$$\rightarrow \beta = 1 - 2\left(\frac{2m_ec^2}{E}\right)^2$$

5. This is a problem treating the concept of "Compton Scattering". An electron of energy $E = \frac{hc}{\lambda}$ collides with a stationary electron and is scattered at an angle θ to the line of incidence. *Find an expression for the new wavelength of the photon as a function of θ.*

As usual, let us write the four-vector momenta:

$$\underline{p}_1^\mu = \frac{h}{\lambda}(1,1,0,0)$$
$$\underline{p}_2^\mu = \left(\frac{mc^2}{c},0,0,0\right)$$
$$\underline{p}_1^\mu = \frac{h}{\lambda'}(1,\cos\theta,\sin\theta,0)$$

Again, we have left out the momenta of the final state of the electron as it is inconsequential; we can get rid of it with an appropriate squaring of four-vector elements.

$$\underline{p}_1^\mu + \underline{p}_2^\mu = \underline{p}_3^\mu + \underline{p}_4^\mu$$
$$\underline{p}_1^\mu + \underline{p}_2^\mu - \underline{p}_3^\mu = \underline{p}_4^\mu$$

Squaring both sides and re-arranging:

$$\underline{p}_1^{\mu 2} + \underline{p}_2^{\mu 2} + \underline{p}_3^{\mu 2} + 2\underline{p}_1^\mu \cdot \underline{p}_2^\mu - 2\underline{p}_1^\mu \cdot \underline{p}_3^\mu - 2\underline{p}_2^\mu \cdot \underline{p}_3^\mu = m^2c^2$$
$$m^2c^2 + 2mc\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) - \frac{2h^2}{\lambda\lambda'}(1-\cos\theta) = mc^2c^2$$

Thus, we obtain a final result of:

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta) \tag{3.19}$$

This holds for any value of θ. For example, the wavelength of a back-scattered electron is given by:

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\pi)$$
$$= \lambda + \frac{2h}{mc}$$
$$\rightarrow \Delta\lambda = \frac{2h}{mc}$$

6. Suppose that an electron of energy E collides obliquely with a stationary electron, scattering both at an angle θ with respect to the line of incidence of the two particles. *Find an expression for θ in terms of E and E_o, the rest energy of the electron.*

Before the collision:

$$\underline{p}_1^{\mu} = \left(\frac{E}{c}, p, 0, 0\right)$$
$$\underline{p}_2^{\mu} = \left(\frac{E_o}{c}, 0, 0, 0\right)$$

After the collision, both of the electrons have to have the same energy by the conservation of energy for both bodies (they have the same mass). We can thus write:

$$\underline{p}_3^{\mu} = \left(\frac{E_1'}{c}, p'\cos\theta, p'\sin\theta, 0\right)$$
$$\underline{p}_4^{\mu} = \left(\frac{E_2'}{c}, p'\cos\theta, -p'\sin\theta, 0\right)$$

By the conservation of momentum and energy:

$$\begin{aligned} p &= 2p'\cos\theta \\ p' &= \frac{p}{2\cos\theta} \\ 2E' &= E + E_o \\ E' &= \frac{E+E_o}{2} \end{aligned}$$

We can thus write the final four-momenta as:

$$\underline{p}_{final}^{\mu} = \left(\frac{E_o+E}{2c}, \frac{p}{2}, \pm\frac{p}{2}\tan\theta, 0\right)$$

Using the Lorentz invariant:

$$\begin{aligned} m^2c^4 &= \left(\frac{E+E_o}{2}\right)^2 - \left(\frac{p}{2}\right)^2(\tan^2\theta+1)c^2 \\ &= \left(\frac{E+E_o}{2}\right)^2 - \left(\frac{p}{2}\right)^2 c^2\sec^2\theta \\ 4m^2c^4 &= (E+E_o)^2 - p^2c^2\sec^2\theta \end{aligned}$$

Re-arranging for θ:

$$\begin{aligned} \sec^2\theta &= \frac{E^2+2EE_o+{E_o}^2-4{E_o}^2}{E^2-{E_o}^2} \\ &= \frac{(E+3E_o)(E-E_o)}{(E+E_o)(E-E_o)} \\ \to \cos\theta &= \sqrt{\frac{E+E_o}{E+3E_o}} \end{aligned}$$

3.5 The Relativistic Doppler Effect

When sources of signals (such as light) move towards or away from an observer, the signal is shifted based on the rate of separation of the observer and the source. This is known as the Doppler Shift.

3.5.1 Derivation

Let a source emitting a signal at a period τ be moving away from an observer at a speed v. Using the LT's, the first pulse of the signal is at $t = 0$, $x = 0$, and the second is at $t = \gamma\tau$, $x = v\gamma\tau$ from the frame of reference of the observer. The observed period is thus given by:

$$\begin{aligned}\tau' &= \gamma\tau + \frac{x}{c} \\ &= \gamma\tau + \frac{v\gamma\tau}{c} \\ &= \gamma\tau\left(1 + \frac{v}{c}\right) \\ \rightarrow \tau' &= \tau\sqrt{\frac{1+\beta}{1-\beta}} \qquad (3.20)\end{aligned}$$

Thus, for a receding source, $\tau' > \tau$ (known as a red-shift), and for an approaching source, $\tau' < \tau$ (known as a blue-shift). This can be used to determine the speed at which planets are receding, for example.

3.5.2 Superluminal Motion

Superluminal motion is where the projected position of an object moving obliquely across the field of view onto the plane of the sky appears to move faster than the speed of light. Consider an object emitted from A that moves with a relativistic speed v along a trajectory passing through B, as shown in Figure (3.6). Let the observer be located at D.

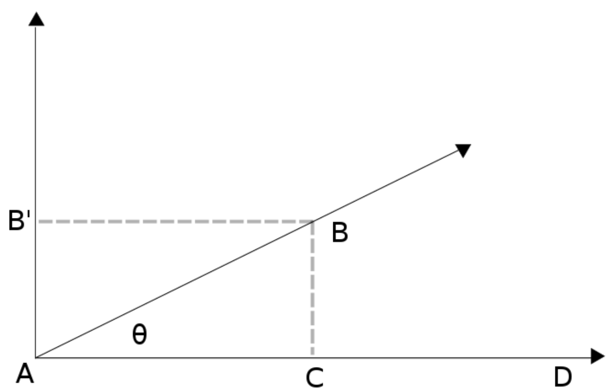

Figure 3.6: Superluminal Motion

We want to calculate the apparent rate of change of AB'.

$$\begin{aligned}AB &= \beta c\delta t \\ AC &= \beta c\delta t\cos\theta \\ AD &= c\delta t\end{aligned}$$

as the emitted photon travels at c. The two photons arriving are separated by:

$$\begin{aligned} DC &= AD - AC \\ &= c\delta t(1 - \beta \cos\theta) \end{aligned}$$

The projected separation of the object is:

$$CB = \beta c \delta t \sin\theta$$

Thus, the apparent speed is given by:

$$\begin{aligned} v_o &= \frac{CB}{DC} \cdot c \\ &= \frac{\beta c \delta t \sin\theta}{c\delta t(1 - \beta\cos\theta)} \\ \rightarrow v_o &= \frac{\beta \sin\theta}{1 - \beta\cos\theta} \end{aligned}$$

Consider an object travelling at $v = 0.9c$ at an angle $\theta = 10°$.

$$\begin{aligned} v_o &= \frac{(0.9)(\cos(10°))}{1 - (0.9)(\cos(10°))} \\ &\approx 1.37\, c \end{aligned}$$

Thus, the projection of the object appears to move faster than c. However, this evidently does not violate causality as no information is actually transmitted at this speed.