CP4: Basic Calculus, Vector Calculus and Waves

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Contents

1. *Basic Calculus*

In this chapter, we cover the main concepts of basic calculus, including:

- Differentiation
- Integration
- The Hyperbolic Functions
- Sequences and Series
- Limits
- Partial Calculus

Most students will be familiar with the majority of the concepts outlined in this chapter. In light of this fact, the chapter does not go into much detail concerning the proofs of many of the theorems; please refer to another text if a more detailed treatment is required.

1.1 Differentiation

As many of you will already know, differentiation allows us to find the rate of change of a function; taking the derivative of a given function $f(x)$ will create another function $f'(x)$ that will give the instantaneous rate of change of the original function. This is used in physics to find the rate of change of certain physical quantities, as well as allowing us to find maxima and minima of functions. We have already encountered both of these ideas in the notes on CP1 and CP2.

1.1.1 Differentiation from First Principles

Consider the two points x and $x + h$ of a function $y = f(x')$, as shown in the diagram below.

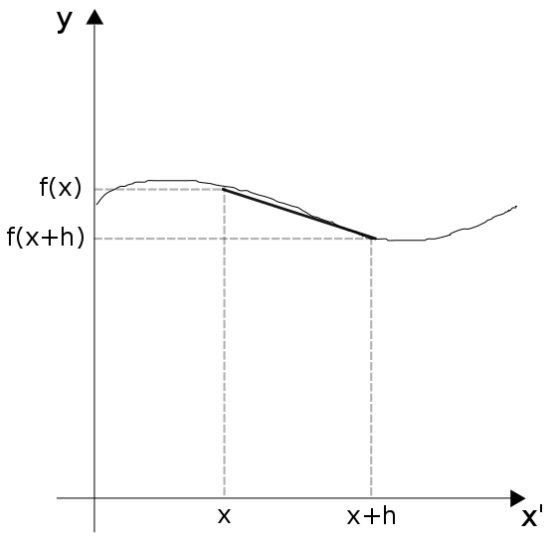

Figure 1.1: Differentiation from first principles

We can write the average gradient over the interval as:

$$\begin{aligned} Gradient &= \frac{\Delta y}{\Delta x'} \\ &= \frac{f(x+h) - f(x)}{x + h - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

Thus, to find the instantaneous gradient, or derivative, we have to take the limit where h is very small. This allows us to obtain the expression:

$$\frac{d}{dx}\left(f(x)\right) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.1}$$

1.1.2 The Chain Rule

Suppose that we have a function $y = f(u)$ where $u = g(x)$. Then we can write:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \tag{1.2}$$

This is known as *the chain rule*. As we will find out later, this is just a specific case of another general property of functions (see (1.21)). It can be used to derive the following expressions that we regularly use in differentiation. The left-hand side corresponds to the function, the right-hand side to it's derivative.

$$\begin{aligned} x^n &\to nx^{n-1} \\ e^{f(x)} &\to f'(x)e^{f(x)} \\ log(f(x)) &\to \frac{f'(x)}{f(x)} \\ \sin x &\to \cos x \\ \cos x &\to -\sin x \\ \tan x &\to \sec^2 x \\ \sec x &\to \sec x \tan x \\ \cot x &\to -\sin^{-2} x \end{aligned}$$

These are always worth memorising such that one does not have to work them out every time that they are to be used.

1.1.3 Properties of the Deritaive

As previously stated, the derivative of a function can tell us a large amount of information about said function.

$f'(x) > 0 \to$ The curve is **rising** (from left to right)
$f'(x) < 0 \to$ The curve is **falling** (from left to right)
$f'(x) = 0 \to$ The curve has a **stationary point**

We can then examine the second-derivative of the function to find the nature of the stationary point, as follows:

$f''(x) < 0 \to$ **Maximum** turning point, concave down
$f''(x) > 0 \to$ **Minimum** turning point, concave up
$f''(x) = 0 \to$ A point of **inflexion**, concavity is changing

These results can become particularly important when examining, for example, the potential energy curve of an object to find it's behaviour.

1.1.4 Leibnitz's Theorem

This is used to find the higher derivatives of a product of two functions. Consider $y(x) = u(x) \cdot v(x)$. The notation of () in the exponentials is used to refer to the order of the derivative of a function.

$$\begin{aligned} y^{(1)} &= u^{(1)}v + uv^{(1)} \\ y^{(2)} &= u^{(2)}v + 2u^{(1)}v^{(1)} + uv^{(2)} \\ y^{(3)} &= u^{(3)}v + 3u^{(2)}v^{(1)} + 3u^{(1)}v^{(2)} + uv^{(3)} \end{aligned}$$

This generalises to:

$$y^n(x) = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots + \binom{n}{n-1}u^{(1)}v^{(n-1)} + uv^{(n)}$$

$$\rightarrow y^n(x) = \sum_{k=0}^{n}\binom{n}{k}u^{(k)}v^{(n-k)} \tag{1.3}$$

where $\binom{n}{k}$ are the binomial coefficients. Always try to choose one of the terms in the product that will disappear so as to reduce the number of derivatives that have to be computed.

1.2 Integration

Integration can be thought of as the opposite of differentiation; instead of finding the rate of change of the curve, it now allows us to find the area between the curve and the x-axis. Physically, we can use this, for example, to find the equation of motion of an object given it's acceleration, as the velocity and displacement are just the time integrals of the acceleration.

1.2.1 The Fundamental Theorem of Calculus

Consider the following function:

$$I(x) = \int_a^x f(x')\,dx'$$

We want to look at the upper and lower rectangles bounding the function, as shown Figure (1.2). From the graph:

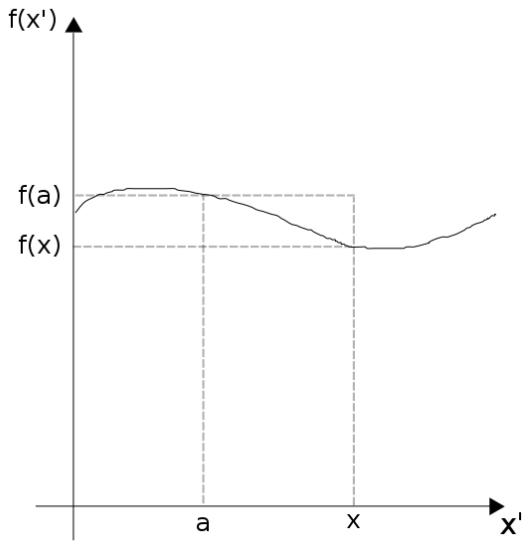

Figure 1.2: Deriving the fundamental theorem of calculus

$$(x-a)\,f(x) \leqslant I(x) \leqslant (x-a)\,f(a)$$

Differentiate both sides and take the limit as $a \to x$.

$$\lim_{a\to x}\Big|\, f(x) + f'(x)\,(x-a) \leqslant \frac{d}{dx}(I(x)) \leqslant \lim_{a\to x}\Big|\, f(a) + f'(a)\,(x-a)$$

Both the left-hand and right-hand sides of this inequality tend to the same value, and so we arrive at the *Fundamental Theorem of Calculus.*

$$\frac{d}{dx}(I(x)) = f(x) \tag{1.4}$$

i.e that the derivative of the integral of function is the function itself.

1.2.2 Standard Integration Techniques

We can use a variety of techniques in order to evaluate a definite (one containing limits of integration) or indefinite integral. In the latter case, always remember to include the constant of integration (usually denoted by C).

- Standard Form - Some integrals we just have to know, as unlike differentiation we cannot easily work out the result from first principles.

$$\int \cos x \, dx = \sin x + \mathrm{C}$$
$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + \mathrm{C}$$

- By Inspection - This is just a sophisticated way of saying "guess-and-check"; we need to check whether the integrand (the expression inside the integral) contains a function and the derivative of a functions argument.

$$\int x e^{x^2} \, dx = \frac{e^{x^2}}{2} + \mathrm{C}$$

- Change of variables - We can make a substation in the integrand to change to a new set of variables that we can more easily use to evaluate the integral. If the integral is definite, remember to change the variables of integration.

$$I = \int \frac{dx}{\sqrt{1-x^2}}$$
$$x = \sin\theta$$
$$dx = \cos\theta \cdot d\theta$$

  Substituting these results into the integral:

$$\begin{aligned} I &= \int \frac{\cos\theta}{\cos\theta} \, d\theta \\ &= \int d\theta \\ &= \theta + \mathrm{C} \\ &= \sin^{-1} x + \mathrm{C} \end{aligned}$$

- Integrals of Sines and Cosines - When the integrand purely contains sines and cosines, always remember to apply the following identities appropriately:

$$\cos^2 x + \sin^2 x = 1 \tag{1.5}$$
$$\tan^2 x + 1 = \sec^2 x \tag{1.6}$$
$$\sin 2x = 2\cos x \sin x \tag{1.7}$$
$$\cos 2x = \cos^2 x - \sin^2 x \tag{1.8}$$

- Integration by Parts - This is essentially the reverse of the product rule for differentiation.

$$\int \frac{d}{dx}(uv) = \int \left(\frac{du}{dx}v + \frac{dv}{dx}u\right) dx$$
$$uv = \int v\frac{du}{dx} \, dx + \int u\frac{dv}{dx} \, dx$$

$$\int u\,dv = uv - \int v\,du \tag{1.9}$$

With an appropriate choice of the functions u and v we can evaluate some more difficult integrals. Consider the following example.

$$I_n = \int_0^\infty x^n e^{-x}\,dx$$

Making the substitution:

$$\begin{aligned} u &= x^n \\ u' &= nx^{n-1} \\ v' &= e^{-x} \\ v &= -e^{-x} \end{aligned}$$

Thus:

$$\begin{aligned} I_n &= \left[x^n e^{-x}\right]_0^\infty + \int_0^\infty nx^{n-1}e^{-x}\,dx \\ &= n\int_0^\infty x^{n-1}e^{-x}\,dx \\ \rightarrow I_n &= n\,I_{n-1} \end{aligned}$$

- Partial Fractions - This involves splitting up a fraction with factored terms in the denominator into individual fractions that can then be integrated. As a general rule, we write out the desired fractions to a coefficient, and then solve for the coefficients by equating powers on either side of the equation. Always remember that if there are quadratic or higher terms in the denominator, we need to include terms up to the order below in the numerator. For example:

$$\begin{aligned} \frac{3x^4+4x^3}{(x+2)\left(x^2+3\right)^2} &= \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{\left(x^2+3\right)^2} \\ 3x^4+4x^3 &= A\left(x^2+3\right)^2 + (Bx+C)\left(x^2+3\right)(x+2) + (Dx+E)(x+2) \end{aligned}$$

and then equate coefficients.

1.2.3 Properties of Definite Integrals

Below are some of the properties of definite integrals which can often by used to simplify some problems.

- For a well behaved function $f(x)$:

$$\int_a^a f(x)\,dx = 0 \tag{1.10}$$

This is intuitively obvious; if we integrate over a zero integral, we should get a zero result.

- Assuming that b lies within the interval $a < x < c$:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

- Letting $c = a$ within the previous identity:

$$\int_a^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)dx \tag{1.11}$$

- If $f(x)$ is even:

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \tag{1.12}$$

  If $f(x)$ is odd:

$$\int_{-a}^{a} f(x)\,dx = 0 \tag{1.13}$$

  This last result follows trivially from applying (1.11).

1.3 Hyperbolic Functions

The *Hyperbolic Functions* are another class of functions (like the hyperbolic functions) that some readers may have already encountered. It is worth learning the properties of these functions and various related identities, but they can be worked out by hand if required. Graphically, they are represented as:

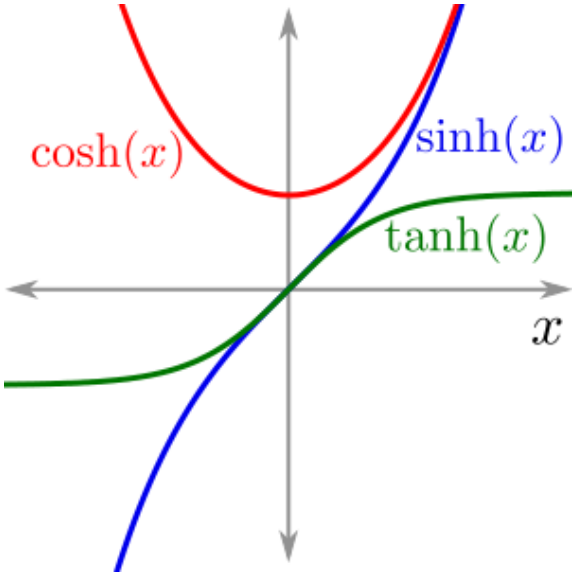

Figure 1.3: The Hyperbolic Functions

1.3.1 Hyperbolic Sine

$$\sinh x = \frac{e^2 - e^{-x}}{2}$$
$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\int \sinh ax \, dx = \frac{\cosh ax}{a}$$
$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

1.3.2 Hyperbolic Cosine

$$\sinh x = \frac{e^2 + e^{-x}}{2}$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\int \cosh ax \, dx = \frac{\sinh ax}{a}$$
$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

1.3.3 Hyperbolic Tangent

$$\begin{aligned}
\tanh x &= \frac{1 - e^{-2x}}{1 + e^{\,2x}} \\
\frac{d}{dx}(\tanh x) &= \cosh^{-2} x \\
\int \tanh x \, dx &= \frac{\ln(\cosh ax)}{a} \\
\tanh -1x &= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)
\end{aligned}$$

1.4 Sequences and Series

In general, sequences are a set of numbers or expressions that are related by an expressible rule, or set of rules. There are two main types of series.

1.4.1 Geometric Series

A geometric series is characterised by a *common ratio*; the series progresses by multiplying the previous term by said common ratio.

$$a + ar + ar^2 + ar^3 + \cdots + ar^n = \sum_{k=0}^{n} ar^k$$

In the case for $|r| > 1$, the sum over all the terms in the series is given by:

$$S = \frac{a\,(1 - r^n)}{1 - r} \tag{1.14}$$

This series is divergent; that is, it will continuously increase in magnitude. However, when $|r| < 1$, the series will converge to a finite value:

$$S_\infty = \lim_{n \to \infty} \frac{a\,(1 - r^n)}{1 - r}$$

$$S_\infty = \frac{a}{1 - r} \tag{1.15}$$

1.4.2 Arithmetic Series

This is characterised by a *common difference*; the series progresses by adding the common difference to the previous term.

$$a + (a + d) + (a + 2d) + \cdots + (a + nd) = \sum_{k=0}^{n} a + (k - 1)d$$

There are no cases of d in which the series is convergent, it will always diverge. The sum of such a series is given by:

$$S = \frac{n}{2}\,(2a + (n - 1)d) = \frac{n}{2}(1^{st}\text{ term} + \text{last term}) \tag{1.16}$$

1.4.3 Taylor Series Expansion

The *Taylor Series Expansion*, more usually referred to as the Taylor Series, is used to approximate the value of a function around a point. Consider the function $y = f(x)$ at the point $x = x_o$. Evidently, the very first approximation to the value of the function at this point is $y = a$. The next better approximation is evidently:

$$f(x) = f(a) + f'(a)(x - a)$$

Then:

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2$$

If we continue extrapolating the series, we arrive at the Taylor series expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(x - a)^n \tag{1.17}$$

We can use this around a point to find the behaviour of an object for small oscillations (we have seen this in the Lagrangian section of the CP1 notes), or just to find approximate expressions for functions given a small parameter. This is one of the bread-and-butter tools for a physicist, and so it is recommended that readers get very familiar with computing and manipulating these expansion expressions.

Often, it can be laborious to calculate these expansions on the fly, and so it is worth remembering a few common ones:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\
\tan x &= x + \frac{x^3}{3!} + \frac{2x^5}{15} + \dots \\
\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots \\
\frac{1}{\sqrt{1+x}} &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots
\end{aligned}$$

1.4.4 The Parametric Integral

We can now combine our knowledge of the Taylor Series and the Fundamental Theorem of Calculus (1.4) to tackle a parametric integral. Consider the following:

$$I(x) = \int_0^x F(x,t)\,dt$$

for a well defined function $F(x,t)$ over the interval of integration. We now want to consider integration from first principles in order to find the derivative of $I(x)$.

$$\begin{aligned}
I(x+dx) - I(x) &= \int_0^{x+dx} F(x+dx,t)\,dt - \int_0^x F(x,t)\,dt \\
&= \int_0^{x+dx} F(x,t)\,dt - \int_0^x F(x,t)\,dt + dx \int_0^{x+dx} F'(x,t)\,dt
\end{aligned}$$

Hence, we arrive at the result that:

$$\frac{dI}{dx} = F(x,x) + \int_0^x F'(x,t)\,dt \tag{1.18}$$

This can be useful when trying to differentiate a 'parametric integral' such as $I(x)$.

1.5 Limits

If the limit of a function exists, the function approaches that value as the argument approaches some limit. The limit exists iff the limit approaching from above and below is the same. Some methods of finding limits include:

- Factoring/conjugate manipulation
- Divide throughout by the highest/lowest power of x in the denominator/numerator to allow terms to disappear
- "Exponentials beat powers beat logs" as the argument tends to some limit
- Expand the function using a Taylor Series on the individual terms to either leading or second order and look at the resultant coefficients. For example:

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1}{\sin x} &= \lim_{x\to 0}\frac{1+x+\frac{x^2}{2!}+\cdots-1}{\sin x}\\ &\approx \lim_{x\to 0}\frac{x}{\sin x}\\ &= 1\end{aligned}$$

1.5.1 L'Hôpital's Rule

This rule states that if

$$\lim_{x\to z}\frac{a(x)}{b(x)}$$

is undefined for some indeterminate form, then the limit is given by:

$$\lim_{x\to z} f(x) = \lim_{x\to z}\frac{a^{(n)}(x)}{b^{(n)}(x)} \tag{1.19}$$

where n is the order of the derivative required for the limit to become defined. For example:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x + x}{x+x^2} &= \lim_{x\to 0}\frac{\cos x+1}{2x+1}\\ &= 2\end{aligned}$$

But you may ask, *where does this come from?* Consider the Taylor expansion of the function in the limit to second order:

$$\begin{aligned}\lim_{x\to z}\frac{a(x)}{b(x)} &= \lim_{x\to z}\frac{a(z)+a'(z)(x-z)+\ldots}{b(z)+b'(z)(x-z)+\ldots}\\ &= \lim_{x\to z}\frac{a(z)}{b(z)}\end{aligned}$$

unless $a(z)=0$, $b(z)=0$. In this case, we divide through by $(x-z)$, and the result follows. This holds true for all derivatives of the quotient until the limit becomes defined.

1.6 Partial Calculus

Thus far, we have only dealt with single variable functions, where derivatives are *total*; that is, they are defined entirely in terms of this single variable. So *how do we take a derivative of a multi-variable function?* In most cases, we cannot find the total derivative of the function as it is not defined in terms of a single variable; we instead need to find the partial derivative.

The partial derivative is found by differentiating the function with respect to the chosen variable *while treating the other as a constant.* Instead of a 'straight' d symbol, we use the symbol ∂ to notate the partial derivative. For example:

$$\begin{aligned} z &= f(x,y) \\ &= 3(x^2+y^2) + e^{-(x^2+y^2)} \\ \frac{\partial z}{\partial x} &= 6x - 2xe^{-(x^2+y^2)} \\ \frac{\partial z}{\partial y} &= 6y - 2ye^{-(x^2+y^2)} \end{aligned}$$

Astute readers will notice that there is some kind of symmetry in this function; in fact, it is radially symmetric, which we can see by letting $r^2 = x^2 + y^2$. This will become more second nature when we cover more coordinate systems in the next chapter.

As it becomes quite tiresome to write out partial derivatives in the form $\frac{\partial f}{\partial x}$ when we are using them a lot, we instead use the notation f_x ; i.e the first partial derivative with respect to x. Similarly, f_{xx} corresponds to $\frac{\partial^2 f}{\partial x^2}$ and so on.

1.6.1 The Total Derivative

Let us first consider the Taylor Series expansion for a two dimensional function. The proof is not given here, but it should be quite intuitively obvious how it is a generalisation as the single variable Taylor Series. Let us expand the function around the point (a,b), and so we assume all the derivatives are evaluated at this point. Let $(x-a) = \delta x$ and $(y-b) = \delta y$.

$$f(x,y) = f(a,b) + f_x\delta x + f_y\delta y + \frac{1}{2!}\left(f_{xx}\delta x^2 + f_{yy}\delta y^2 + 2f_{xy}\delta x\delta y\right) + \dots \tag{1.20}$$

Now, imagine that we want to obtain an expression for a small change in the function δf.

$$\begin{aligned} \delta f &= f(x+\delta x, y+\delta y) - f(x,y) \\ &= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x,y) \\ &= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\cdot\delta x + \frac{f(x, y+\delta y) - f(x,y)}{\delta y}\cdot\delta y \end{aligned}$$

Taking the limit where $\delta x \to 0$ and $\delta y \to 0$:

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy \tag{1.21}$$

This is known as *the total derivative* of a function, and is a more general version of the chain rule for a single variable function. The subscripts outside the brackets denote the variable that we are holding constant when doing the differentiation. It is not always necessary to put these in when you become more familiar with partial derivatives, but it does

help to keep track of what is being held constant.

This should be the starting point for finding expressions using partial derivatives. Consider a function $f = f(x, y, z)$.

$$dx = \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz$$
$$dy = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial z}dz$$

Substituting one into the other, we obtain:

$$dx = \left(\frac{\partial x}{\partial y}\frac{\partial y}{\partial x}\right)dx + \left(\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial x}{\partial z}\right)dz$$

If we hold z constant (i.e $dz = 0$, the change is z is zero), we get the *reciprocity relation*:

$$\frac{\partial x}{\partial y} = \left(\frac{\partial y}{\partial x}\right)^{-1} \tag{1.22}$$

If we hold x constant, we get the *cyclical relation*:

$$\frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} = -1 \tag{1.23}$$

The second property may be less intuitively obvious than the first, but both are important properties of partial derivatives worth bearing in mind.

A particular class of functions, called *exact differentials*, satisfy the property that their derivatives commute; the second successive partial derivatives are the same.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{1.24}$$

We deal with the properties of exact differentials fully in the next chapter.

1.6.2 Stationary Points and Contour Sketches

Like with single variable functions, multi-variable functions also have stationary points, though the conditions for them to exist is a little more complicated. To understand this, let us first introduce the concept of a contour sketch.

A contour sketch is a way of representing the form of a multi-variable function in a two-dimensional format, as evidently $z = f(x, y)$ represents a three-dimensional surface (z is the 'height' of the function). Essentially, we can think of a contour sketch as being like a geographical contour map; it tells us where the function increases and decreases, and the rapidity with which it does so. Some general forms of the contours are shown below.

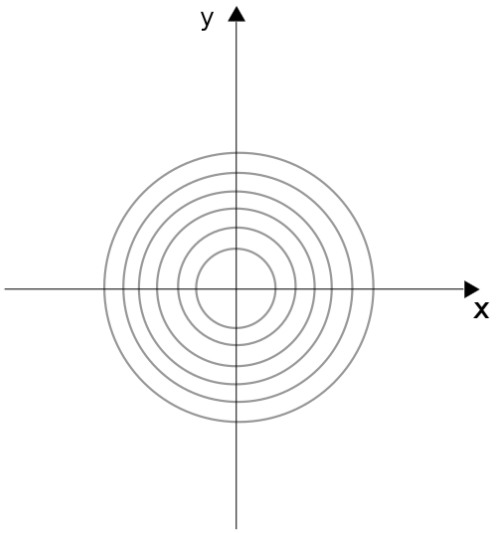

Figure 1.4: The contours around a minimum or a maximum

The contours around a minimum and maximum are closed curves in the forms of ellipses or circles. We show whether the function is increasing or decreasing with a gradient arrow.

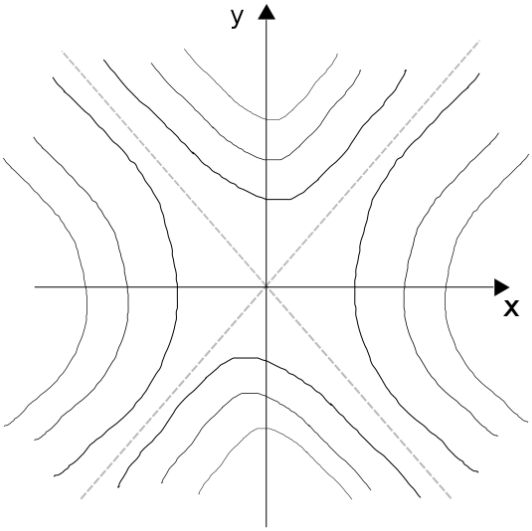

Figure 1.5: The contours around a saddle point

The contours around a circle are open hyperbolae, bounded by *level lines* (shown with the dotted lines). *But where does this come from?*

In a similar vein to the single variable function, for a function $f = f(x, y)$, at a stationary point we require:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{1.25}$$

This is equivalent to saying that the derivative is stationary in both the x and y directions. The nature of the stationary points can be determined by looking at the change in the

function $df = f(x,y) - f(a,b)$ around the stationary point (a,b) under the condition given by (1.25).

$$\begin{aligned} 0 &= f(x,y) - f(a,b) \\ &= f_{xx}(x-a)^2 + f_{yy}(y-b)^2 + 2f_{xy}(x-a)(y-b) \\ &= f_{xx}\,\delta x^2 + f_{yy}\,\delta y^2 + 2f_{xy}\,\delta x\delta y \end{aligned}$$

We can write this equation as a quadratic form by letting $\delta\underline{r} = (\delta x,\ \delta y)$.

$$\begin{aligned} 0 &= \delta\underline{r}\cdot\mathbf{A}\cdot\delta\underline{r} \\ \mathbf{A} &= \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \end{aligned}$$

For a solution to exist, we require that $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$:

$$\begin{aligned} 0 &= \lambda^2 - (f_{xx} + f_{yy})\lambda + f_{xx}f_{yy} - {f_{xy}}^2 \\ \lambda &= \frac{1}{2}\left(f_{xx} + f_{yy} \pm \sqrt{(f_{xx}+f_{yy})^2 - 4(f_{xx}\,f_{yy} - {f_{xy}}^2)}\right) \end{aligned}$$

As we have seen above, the contour lines around a maximum or minimum have elliptical level lines; this means that we need the eigenvalues λ_1 and λ_2 to have the same sign. For a saddle point, the contour lines are hyperbolae, meaning that the eigenvalues have to have opposite signs. In this case, the gradient of the level lines is given by:

$$\eta = \pm\frac{\lambda_1}{|\lambda_2|} \tag{1.26}$$

This gives rise to the following conditions:

- At a maximum:

$$\begin{aligned} {f_{xy}}^2 &< f_{xx}\,f_{yy} \\ f_{xx} &< 0 \\ f_{yy} &< 0 \end{aligned}$$

- At a minimum:

$$\begin{aligned} {f_{xy}}^2 &< f_{xx}\,f_{yy} \\ f_{xx} &> 0 \\ f_{yy} &> 0 \end{aligned}$$

- At a saddle point:

$$ {f_{xy}}^2 \geqslant f_{xx}\,f_{yy} $$

If you are confused about this derivation in that it uses the concept of quadratic forms, we would recommend that you take a look at the section on quadratic forms in the CP3 notes.

2. *Vector Calculus*

This chapter aims to cover the basic concepts of vector calculus, including:

- Coordinate Systems
- Multiple Integrals
- Surface, Volume and Flux Integrals
- The Del (∇) Operator
- Index Notation and Vector Calculus
- Vector Calculus Theorems
- Line Integrals and Conservative Vector Fields

Vector Calculus is what the name suggests; it is the calculus of vectors and vector fields. This is a very useful tool in many areas of physics, but for this course is mainly used in our treatment of Electromagnetism. Some of the concepts covered in this section might seem initially difficult to grasp, but they do become a lot easier with practise and familiarity. Some examples have been included to illustrate the concepts introduced.

2.1 Coordinate Systems

Thus far, we have mainly dealt with the Cartesian and plane-polar coordinate systems. However, there are other coordinate systems which are commonly used, and it is imperative that any physics student is familiar with these. Do not be alarmed if you are not familiar with some of the terms here; they will become clear as you read on, I have just placed them here for reference.

2.1.1 Cartesian Coordinates

The unit vectors are $\underline{\hat{x}}$, $\underline{\hat{y}}$ and $\underline{\hat{z}}$. As previously discussed, these have time-invariant directions. The coordinate system has the following characteristics:

- Line element: $ds^2 = dx^2 + dy^2 + dz^2$
- Area element: $d\underline{S} = (dxdy,\ dxdz,\ dydz)$
- Volume element: $dV = dxdydz$

2.1.2 Cylindrical Coordinates

As the name suggests, these are used when there is some kind of cylindrical symmetry to the object, such as when there is plane-polar symmetry and a variation in the $\underline{\hat{z}}$ direction. The unit vectors are $\underline{\hat{r}}$, $\underline{\hat{\theta}}$, and $\underline{\hat{z}}$.

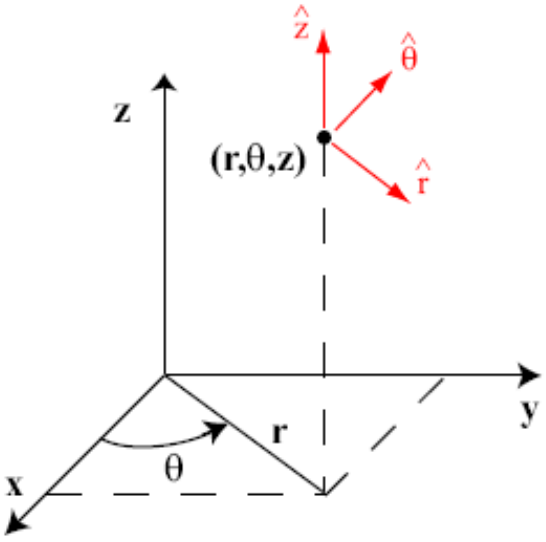

Figure 2.1: Cylindrical Coordinates

Converting from Cartesian coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

This coordinate system has the following characteristics:

- Line element: $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

- Area element: Let a be the radius for some planar circular area parallel with the xy plane

$$d\underline{S} = \hat{\underline{r}} \cdot a^2 \, d\theta dz$$
$$\hat{\underline{r}} = (\cos\theta, \, \sin\theta, \, 0)$$

- Volume element: $dV = r \, dr d\theta dz$

As we have already seen with plane-polar coordinates, the $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ unit vectors are not constant in time for a moving point, though $\hat{\underline{z}}$ is.

2.1.3 Spherical Coordinates

As the name suggests, these are used when there is some kind of spherical symmetry to the object, such as when the field is entirely radial in the case of a central force. The unit vectors are $\hat{\underline{r}}$, $\hat{\underline{\phi}}$, and $\hat{\underline{\theta}}$.

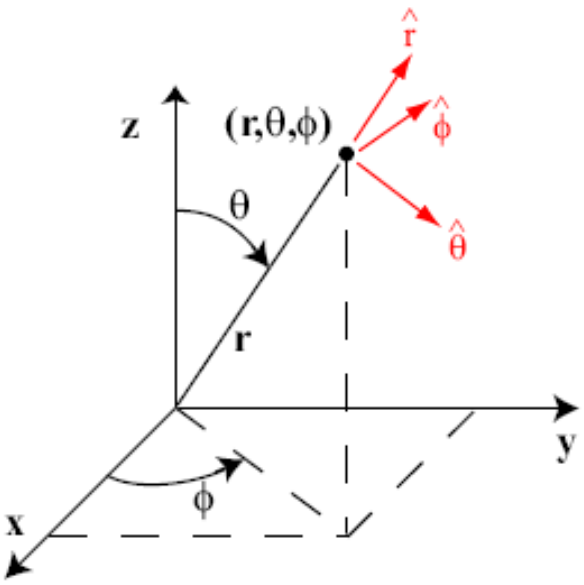

Figure 2.2: Spherical Coordinates

Converting from Cartesian coordinates:

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

The coordinate system has the following characteristics:

- Line element: $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$
- Area Element: Let a be the value of the radius for some sphere

$$d\underline{S} = \hat{\underline{r}} \cdot a^2 \sin\theta \, d\theta d\phi$$
$$\hat{\underline{r}} = (\cos\phi\sin\theta, \, \sin\phi\sin\theta, \, \cos\theta)$$

- Volume element: $dV = r^2 \sin\theta \, d\theta d\phi dr$

2.2 Multiple Integrals

These are similar to normal integrals, except we instead of performing integration over a single 'element', such as dx, we instead integrate successively over a series of elements. To get an idea of how this is done, consider the following example.

Evaluate the integral

$$I = \int\int_A x\, dxdy$$

where A is the finite area enclosed by the parabola $y = x^2$ and the straight line $2x-y+8 = 0$.

First, let us sketch the area of integration:

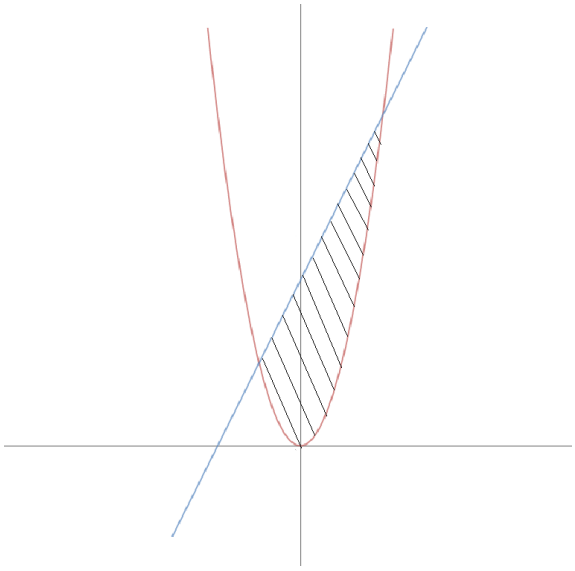

Figure 2.3: A basic example

The two curves intersect at:

$$\begin{aligned} y &= 2x + 8 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ x &= -2 \text{ or } 4 \end{aligned}$$

At this stage, we need to choose which order we want to integrate in; dx then dy or dy the dx. In this case, seeing as we already have the dx *numerical* limits, we will do these second, as follows:

$$\begin{aligned} I &= \int_{-2}^{4} x\, dx \int_{x^2}^{2x+8} dy \\ &= \int_{-2}^{4} x\, dx\, [y]_{x^2}^{2x+8} \\ &= \int_{-2}^{4} dx\, 2x^2 + 8x - x^3 \\ &= \left[\frac{2x^3}{3} + 4x^2 - \frac{x^4}{4}\right]_{-2}^{4} \\ \rightarrow I &= 36 \end{aligned}$$

Notice how the limits on the first integral actually contained the variable x. This is because the limits on the dy integral, instead of being from a value to a value, are instead from a curve to a curve. In this case, we integrate from the lower curve $y = x^2$ to the upper curve $y = 2x + 8$. It is recommended to get some practise at doing these types of integrals to get used to picking the limits.

2.2.1 Changing the Order of Integration

The order in which the integration can be carried out can be reversed in order to simplify the integral. First, sketch the region of integration. For the outer bounds (i.e those that we will compute first), draw a line from $-\infty$ to ∞ in the direction required. The order that the line crosses the curves gives the order of the first limits. Then the second limits are just the numerical bounds that correspond to these limits.

Evaluate the integral

$$I = \int_0^\pi dy \int_y^\pi dx \, \frac{\sin x}{x}$$

by changing the order of integration.
Again, we start by sketching the region of integration:

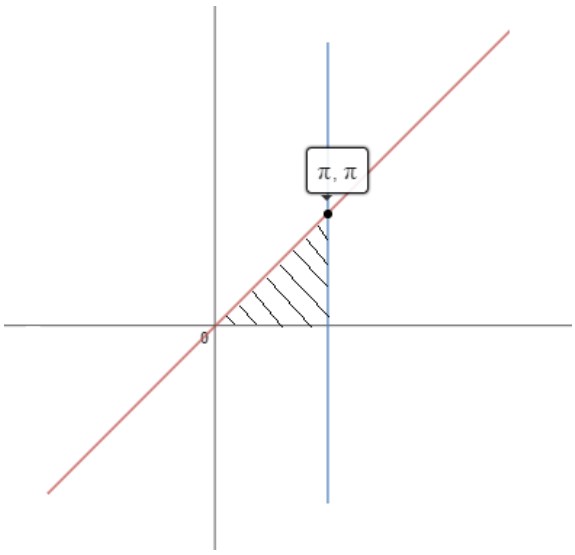

Figure 2.4: Changing the order of integration

Applying the technique described above, it becomes clear that:

$$dx : 0 \to x$$
$$dy : 0 \to \pi$$

Thus, we can more easily evaluate the integral:

$$\begin{aligned} I &= \int_0^\pi dx \int_0^x \frac{\sin x}{x} \, dy \\ &= \int_0^\pi dx \, \sin x \\ \to I &= 2 \end{aligned}$$

2.2.2 Change of Basis

A coordinate or basis transformation can be used to simply multiple integrals when the region of integration is particularly complicated, or the integrand itself is particularly complicated. We do this using a quantity called the Jacobian.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{2.1}$$

where u and v are the new basis coordinates, such as r or θ. This has the important property that:

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1 \tag{2.2}$$

This can be proven by using the property that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for two matrices $\mathbf{A}$ and $\mathbf{B}$.

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial x}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, $\det(\mathbf{AB}) = 1$ and so the result follows. Though we have just considered the 'two-variable' case to derive this result, it actually holds for the n-dimensional Jacobian.

Evaluate the following integral

$$I = \int\int_A dxdy \ \left(x^2 - y^2\right) \ e^{(x^2 - xy + y^2)}$$

in the region bounded by $y = x$, $y = x - 1$, $y = \frac{1}{x}$ *and* $x = 0$, *using the transformation* $u = x - y$ *and* $v = xy$.

The trick to these questions is finding the new limits of integration; the rest is just trivial manipulation of the integrand.

$$\begin{aligned} y &= x - 1 \\ \rightarrow u &= 1 \end{aligned}$$

$$\begin{aligned} y &= x \\ \rightarrow u &= 0 \end{aligned}$$

$$\begin{aligned} y &= \frac{1}{x} \\ \rightarrow v &= 1 \end{aligned}$$

$$\begin{aligned} y &= 0 \\ \rightarrow v &= 0 \end{aligned}$$

Now, computing the Jacobian:

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} 1 & -1 \\ y & x \end{vmatrix} \\ &= x + y\end{aligned}$$

Using (2.2), it is then clear that:

$$dxdy = \frac{1}{x+y}\, dudv$$

We can now use all this information to evaluate the integral.

$$\begin{aligned} I &= \int\int_A dxdy\ \left(x^2 - y^2\right)\ e^{(x^2 - xy + y^2)} \\ &= \int\int_A dxdy\ (x+y)(x-y)\, e^{(x-y)^2 + xy} \\ &= \int\int_{A'} dudv\ u\, e^{u^2+v} \\ &= \int_0^1 du \int_0^1 dv\, e^{u^2+v} \\ \rightarrow I &= \frac{1}{2}(e-1)^2 \end{aligned}$$

2.3 Surface and Volume Integrals

In order to compute a volume or surface area, we require expressions for dS or dV and the appropriate limits of integration.

$$S = \int dS \tag{2.3}$$

$$V = \int dV \tag{2.4}$$

To illustrate this, consider the following simple example.

A sphere of radius R is centred on the origin. Find the volume enclosed by the two parallel planes at $z = \pm a$.

Let us first, as always, find the limits of integration. Incidentally, this is always the hardest bit of these types of questions, as long as you can integrate properly! We can use cylindrical polar coordinates.

$$\begin{aligned} dr &: 0 \to \sqrt{R^2 - z^2} \\ d\theta &: 0 \to 2\pi \\ dz &: -a \to a \end{aligned}$$

Hence, the volume is given by:

$$\begin{aligned} V &= \int\int\int_V r\, dr d\theta dz \\ &= \int_{-a}^{a} dz \int_0^{2\pi} d\theta \int_0^{\sqrt{R^2-z^2}} dr\, r \\ \to V &= \frac{2\pi a}{3}(3R^2 - a^2) \end{aligned}$$

A quick note; a cone is generally defined by the equation:

$$r = a\left(1 - \frac{z}{h}\right) \tag{2.5}$$

We can also use similar techniques to find the mass and centre of mass of a particular laminar surface or volume:

$$M = \int dm = \int \rho\, dV \tag{2.6}$$

$$\underline{r}_{CM} = \frac{1}{M}\int \underline{r}\, dm \tag{2.7}$$

2.3.1 Coordinate Transformations

As we have seen before, coordinate transformations can be useful in evaluating surface or volume integrals if there is some kind of symmetry that the problem can be reduced to.

Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = R^2$$

At first glance, this problem looks terribly complicated. However, let us first start by making the following substitutions:

$$u = \frac{x}{a},\ v = \frac{y}{b},\ w = \frac{z}{c}$$

We obtain

$$u^2 + v^2 + w^2 = R^2$$

which is the equation for a sphere in the uvw coordinate system. If we compute the Jacobian for this transformation, we can then easily evaluate the integral, as we know how to find the volume of a sphere.

$$\begin{aligned} dxdydz &= abcdudvdw \\ &= abc\, r^2 \sin\theta\, drd\theta d\phi \\ V &= \int dxdydz \\ &= adc \int r^2 \sin\theta\, drd\theta d\phi \\ \rightarrow V &= \frac{4\pi}{3} abc\, R^3 \end{aligned}$$

2.3.2 The Method of Projection

The method of projection, also known as *Monge Projection*, allows us to integrate over a surface defined by $z = f(x, y)$. Let $\widehat{\underline{n}}$ be the unit normal to the surface, and $\widehat{\underline{k}}$ be the normal z unit vector, as shown below.

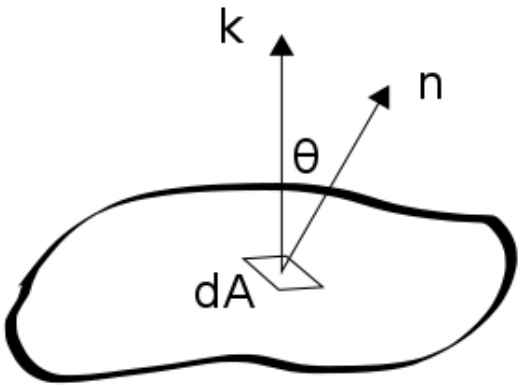

Figure 2.5: Considering a surface element dA

As $\widehat{\underline{n}}$ and $\widehat{\underline{k}}$ are unit vectors:

$$\begin{aligned} \widehat{\underline{n}} \cdot \widehat{\underline{k}} &= \cos\theta \\ dxdy &= \cos\theta\, dS \\ dS &= \frac{1}{|\widehat{\underline{n}} \cdot \widehat{\underline{k}}|}\, dxdy \end{aligned}$$

Hence:

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy \tag{2.8}$$

As usual, let us consider an example to illustrate this method. *Find the volume under the inverted paraboloid $z = 6 - x^2 - y^2$ and above $z = 0$, as well as it's surface area.*

First, the volume (this is usually the easier calculation):

$$\begin{aligned} V &= \int\int\int_V r\, dr d\theta dz \\ &= \int_0^{2\pi} d\theta \int_{\sqrt{6}}^0 dr \int_0^{6-r^2} r\, dz \\ \rightarrow V &= 18\pi \end{aligned}$$

Now for the surface area:

$$\begin{aligned} dS &= \sqrt{1 + (-2x)^2 + (-2y)^2}\, dxdy \\ &= \sqrt{1 + 4(x^2 + y^2)}\, dxdy \\ &= \sqrt{1 + 4r^2}\, dxdy \\ \rightarrow dS &= \sqrt{1 + 4r^2}\, r\, dr d\theta \end{aligned}$$

Putting it all together:

$$\begin{aligned} S &= \int\int_S \sqrt{1 + 4r^2}\, r\, dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{6}} \sqrt{1 + 4r^2}\, r\, dr \\ \rightarrow S &= \frac{31\pi}{12} \end{aligned}$$

Now, find the volume enclosed by the surfaces $z = ax^2 + by^2$ and $z = 6 - cx^2 - dy^2$.

These two curves intersect along the surface $x^s(a + c) + y^2(b + d) = 6$. The limits on the integration are thus:

$$\begin{aligned} dx &: -\frac{\sqrt{6}}{\sqrt{a+c}} \rightarrow \frac{\sqrt{6}}{\sqrt{a+c}} \\ dy &: -\frac{\sqrt{6}}{\sqrt{b+d}} \rightarrow \frac{\sqrt{6}}{\sqrt{b+d}} \end{aligned}$$

We can then make the substitutions that:

$$dy = \frac{dy'}{\sqrt{b+d}},\ dx = \frac{dx'}{\sqrt{a+c}}$$

Hence, using cylindrical coordinates as before:

$$\begin{aligned} V &= \int_{-6}^{6} \frac{dx'}{\sqrt{a+c}} \int_{-6}^{6} \frac{dy'}{\sqrt{b+d}} \int_0^{6-r'^2} dz \\ &= \frac{1}{\sqrt{(a+c)(b+d)}} \int_0^6 dr' \int_0^{2\pi} d\theta \int_0^{6-r'^2} dr' \\ \rightarrow V &= \frac{18\pi}{\sqrt{(a+c)(b+d)}} \end{aligned}$$

2.3.3 Flux Integrals

These are integrals that allow us to compute the 'amount' of a vector field passing through a surface. Put in another way, it is a measure of how much the surface 'cuts' the field. Generally, such integrals are of the form:

$$Flux = \int_S \underline{F} \cdot d\underline{S} \tag{2.9}$$

where $d\underline{S} = \widehat{\underline{n}} \cdot dS$. The trick to evaluating these integrals is finding an expression for the unit normal to the surface $\widehat{\underline{n}}$. In general, it can be computed via the following:

$$\underline{n} = \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right| \tag{2.10}$$

where $\underline{r}$ is the general position vector, with $\underline{u}$ and $\underline{v}$ being the other variables with which $\underline{r}$ is parametrised. Let's do an example.

Calculate

$$\int \underline{F} \cdot d\underline{S}$$

for $\underline{F} = (6z, 2x + y, -x)$ *the region bounded by* $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ *and* $y = 8$.

We can parametrise the surface easily as it has cylindrical symmetry ($x^2 + z^2 = 9$ represents an infinite cylinder with radius 3 with axis in the $\widehat{\underline{y}}$ direction). Let $y = t$, $x = 3\cos\theta$ and $z = 3\sin\theta$.

$$\begin{aligned} r(\theta, t) &= (3\cos\theta,\ t,\ 3\sin\theta) \\ \frac{\partial r}{\partial \theta} &= (-3\sin\theta,\ 0,\ 3\cos\theta) \\ \frac{\partial r}{\partial t} &= (0,\ 1,\ 0) \\ \widehat{\underline{n}} &= (\cos\theta,\ 0,\ \sin\theta) \end{aligned}$$

Note we have normalised the normal vector here. The rest of the evaluation has been left for the reader to complete, but they should find that $Flux = 18\pi$.

2.4 The Del (∇) Operator

The Del Operator, notated by ∇, is a 'derivative operator', given by:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{2.11}$$

This is used in Vector Calculus to perform three main operations.

2.4.1 Gradient

This operation creates a vector from a scalar field or potential. Essentially, it tells us how a scalar field manifests itself as a vector field. We have already seen this in terms of the force resulting from a conservative potential; $\underline{F} = -\nabla U$.

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \tag{2.12}$$

Taking the gradient of a scalar function ϕ will generate a vector normal to the surface described by ϕ. Consider a curve $\underline{r}(t)$ that lies on ϕ.

$$d\phi = \nabla\phi \cdot d\underline{r}$$

For $d\phi = 0$, $\nabla\phi \cdot d\underline{r} = 0$. However, for this to be the case, $d\underline{r}$ must lie along the curve. This means that:

$$\rightarrow \nabla\phi \perp d\underline{r}$$

A successive application of the gradient then divergence operator is known as the Laplace Operator.

$$\Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.13}$$

This crops up a a lot in the physics of waves, as we have already seen with Electromagnetic Waves in CP2.

2.4.2 Divergence

This can be thought of as a measure of how a vector field "spreads out" from a point or through a region.

$$\nabla \cdot \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{2.14}$$

2.4.3 Curl

This is an analogue for the way a field "rotates", or the amount of vector field sources or sinks in a region. If a field has zero curl, it is said to be *irrotational*.

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{2.15}$$

We compute this in the similar way to the cross-product by evaluating the determinant above.

2.4.4 Directional Derivative

The *Directional Derivative* of a function at a point is given by:

$$\nabla_f \phi = \nabla\phi \cdot \underline{\hat{v}} \tag{2.16}$$

where $\underline{\hat{v}}$ a unit vector in the direction along which you wish to evaluate the derivative. It has a couple of useful properties:

- The greatest change in the function ϕ is along the direction of $\nabla\phi$ as this is perpendicular to the surface
- Consequently, the function is stationary along the tangent plane or line with unit normal given by:
$$\underline{\hat{n}} = \frac{\nabla\phi}{|\nabla\phi|} \tag{2.17}$$
- For a small change in the value of the function $d\phi$, we can approximate it by:
$$d\phi = d\,|\nabla\phi| \tag{2.18}$$
where d is the distance moved in the direction of $\nabla\phi$. Note that this is only defined for small changes in the function as we have assumed that $\nabla\phi$ is constant as we move along the function.

2.5 Index Notation and Vector Calculus

It will be assumed that the reader has already read the section in the CP3 notes on index notation, and so is already familiar with the way it functions. Some important expressions include:

$$\partial_i r_j = \delta_{ij} \tag{2.19}$$
$$\nabla = \partial_i e_i \tag{2.20}$$
$$(\nabla \times \underline{F})_i = \epsilon_{ijk}\partial_j F_k \tag{2.21}$$

Using these, we can prove a variety of scalar and vector identities. As always, index notation remains very much the fastest way to prove a lot of these.

2.5.1 Scalar Identities

We will use ϕ and ψ to denote our scalar fields, and $\underline{F}$ to denote our vector fields.

- $\nabla \times (\nabla\phi) = 0$

$$\begin{aligned}\nabla \times (\nabla\phi) &= \epsilon_{ijk}\partial_i\partial_k\phi \\ &= \epsilon_{ikj}\partial_k\partial_i\phi \\ &= -\epsilon_{ijk}\partial_i\partial_k\phi \\ &= 0\end{aligned}$$

  This is due to the properties of the perfectly anti-symmetric tensor, and the fact that derivatives commute.

- $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

$$\begin{aligned}\nabla(\phi\psi) &= \partial_i(\phi\psi) \\ &= \phi\partial_i\psi + \psi\partial_i\phi \\ &= \phi\nabla\psi + \psi\nabla\phi\end{aligned}$$

  Here, we have simply used the chain rule for the derivative with ∂_i.

- $\nabla \cdot (\phi\underline{F}) = \phi\nabla \cdot \underline{F} + \underline{F}\,\nabla\phi$

$$\begin{aligned}\nabla \cdot (\phi\underline{F}) &= \partial_i(\phi F)_i \\ &= \phi(\partial_i F_i) + F_i(\partial_i\phi) \\ &= \phi\nabla \cdot \underline{F} + \underline{F}\,\nabla\phi\end{aligned}$$

- $\nabla r^n = \underline{r}\, nr^{n-2}$ where r is the magnitude of the position vector. For some scalar function $f(r)$:

$$\begin{aligned}\partial_i f(r) &= f'(r)\partial_i r \\ &= f'(r)\frac{x_i}{r} \\ \nabla f(r) &= \underline{r}\, f'(r)\frac{1}{r} \\ \nabla r^n &= \underline{r}\,\frac{d}{dr}(r^n)\,\frac{1}{r} \\ &= \underline{r}\, nr^{n-2}\end{aligned}$$

  This is a result worth remembering, as it can be very useful when it comes to radial fields.

2.5.2 Vector Identities

In this case, we will use $\underline{F}$ and $\underline{G}$ to denote our vector fields.

- $\nabla \cdot (\underline{F} \times \underline{G}) = \underline{G} \cdot (\nabla \times \underline{F}) - \underline{F} \cdot (\nabla \times \underline{G})$

$$\begin{aligned}(\nabla \cdot (\underline{F} \times \underline{G}))_i &= \partial_i(\epsilon_{ijk} F_j G_k) \\ &= \epsilon_{ijk}(\partial_i F_j) G_k + \epsilon_{ijk} a_j (\partial_i G_k) \\ &= \epsilon_{kij} G_k (\partial_i F_j) - \epsilon_{jik} F_j (\partial_i G_K) \\ &= \underline{G} \cdot (\nabla \times \underline{F}) - \underline{F} \cdot (\nabla \times \underline{G})\end{aligned}$$

- $\nabla \cdot (\nabla \times \underline{F}) = 0$
  This is proven in the same way as the analogue identity for the scalar fields above.

- $\nabla \times (\underline{F} \times \underline{G}) = \underline{F}(\nabla \cdot \underline{G}) - \underline{G}(\nabla \cdot \underline{F}) + (\underline{G} \cdot \nabla)\underline{F} - (\underline{F} \cdot \nabla)\underline{G}$

$$\begin{aligned}\nabla \times (\underline{F} \times \underline{G}) &= \epsilon_{klm} \partial_l (\underline{F} \times \underline{G})_m \\ &= \epsilon_{ijk}\epsilon_{klm} \partial_l F_i G_j \\ &= (\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}) \partial_l F_i G_j \\ &= \partial_j (F_i G_j) - \partial_i (F_i G_j) \\ &= F_i \partial_j G_j + G_j \partial_j F_i - (F_i \partial_j G_j + G_j \partial_i F_i) \\ &= \underline{F}(\nabla \cdot \underline{G}) - \underline{G}(\nabla \cdot \underline{F}) + (\underline{G} \cdot \nabla)\underline{F} - (\underline{F} \cdot \nabla)\underline{G}\end{aligned}$$

- $\nabla \times (\nabla \times \underline{F}) = \nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$
  Let $\underline{G} = \nabla$ in the previous identity, and the result follows.

- $\nabla \cdot (r^n \underline{r}) = r^n (3 + n)$

$$\begin{aligned}\nabla \cdot (r^n \underline{r}) &= \partial_i (r^n \underline{r})_i \\ &= \partial_i r^n r_i \\ &= r_i \partial_i r^n + r^n \partial_i r_i \\ &= r_i r_i n r^{n-2} + r^n \partial_i r_i \\ &= r^n \nabla \cdot \underline{r} + \underline{r} \cdot \nabla r^n \\ &= 3r^n + \underline{r} \cdot n r^{n-2} \underline{r} \\ &= 3r^n + r^2 \, n r^{n-2} \\ &= r^n (3 + n)\end{aligned}$$

- $\nabla \times (r^n \underline{r}) = 0$

$$\begin{aligned}\nabla \times (r^n \underline{r}) &= \epsilon_{ijk} \partial_j (r^n \underline{r})_k \\ &= \epsilon_{ijk} \left(\partial_j (r^n r_k)\right) \\ &= \epsilon_{ijk} (r^n \partial_i r_k + r_k \partial_j r^n) \\ &= \nabla r^n \times \underline{r} + r^n \nabla \times \underline{r} \\ &= \nabla r^n \times \underline{r} \\ &= n r^{n-2} \, \underline{r} \times \underline{r} \\ &= 0\end{aligned}$$

2.5.3 Integral Identities

There are also some identities involving integrals that can be proven quickly using index notation. Note that these proofs make use of the theorems shown in Section (2.6), but they have been included here for the sake of neatness.

- Surface integral over a scalar field:

$$\begin{aligned}\left(\int_{\partial V} \rho \, d\underline{S}\right)_i &= \int_{\partial V} \rho \delta_{ij} \, dS_i \\ &= \int_V \partial_i \rho \delta_{ij} \, dS_i \\ &= \int_V \partial_j \rho \, dV \\ &= \left(\int_V \nabla \rho \, dV\right)_i \\ \rightarrow \int_{\partial V} \rho \, d\underline{S} &= \int_V \nabla \rho \, dV\end{aligned}$$

- Vector surface product:

$$\begin{aligned}\left(\int_{\partial V} d\underline{S} \times \underline{a}\right)_i &= \int_{\partial V} \epsilon_{ijk} dS_j \, a_k \\ &= \int_V \epsilon_{ijk} \partial_j a_k \, dV \\ &= \left(\int \nabla \times \underline{a} \, dV\right)_i \\ \rightarrow \int_{\partial V} d\underline{S} \times \underline{a} &= \int \nabla \times \underline{a} \, dV\end{aligned}$$

- Line integral of scalar field:

$$\begin{aligned}\left(\oint_{\partial S} \phi \, d\underline{l}\right)_m &= \oint_{\partial S} \delta_{im} \phi \, dl_i \\ &= \int_S \epsilon_{ijk} \partial_j \delta_{km} \phi \, dS_i \\ &= \int_S \epsilon_{ijm} \partial_j \phi \, dS_i \\ &= \left(\int d\underline{S} \times \nabla \phi\right)_m \\ \rightarrow \oint_{\partial S} \phi \, d\underline{l} &= \int d\underline{S} \times \nabla \phi\end{aligned}$$

2.6 Vector Calculus Theorems

There are three very useful theorems in vector calculus that can be used to simplify a large variety of calculations.

2.6.1 Gauss' Theorem

Also know as *the Divergence Theorem*, this essentially allows us to turn a flux integral over a surface into a volume integral.

Let V be the volume of space that is bounded by ∂V. Then the total flux of $\underline{F}$ through ∂V is equal to:

$$\oint_{\partial V} \underline{F} \cdot d\underline{S} = \int_V \nabla \cdot \underline{F}\, dV \tag{2.22}$$

This can often make calculations much easier as we do not have to compute the dot product in the original integral.

Verify Gauss' Theorem by finding the flux the following field

$$\underline{F} = \frac{\widehat{r}}{r^\alpha}$$

over two concentric shells of radius r_1 and r_2.

We need to find both sides to (2.22) in this case. From the results in the previous section:

$$\begin{aligned} \nabla \cdot \underline{F} &= \frac{2-\alpha}{r^{\alpha+1}} \\ \int \nabla \cdot \underline{F}\ dV &= \int \frac{2-\alpha}{r^{\alpha+1}}\, r^2\, \sin\theta\, dr d\theta d\phi \\ &= 4\pi \left({r_2}^{2-\alpha} - {r_1}^{2-\alpha} \right) \end{aligned}$$

Now for the flux integral:

$$\begin{aligned} \int \underline{F} \cdot d\underline{S} &= \int \frac{\widehat{r}}{r^\alpha} \cdot \widehat{r}\, r^2 \sin\theta\, d\phi d\theta \\ &= 4\pi \left[r^2 r^{-\alpha} \right]_{r_1}^{r_2} \\ &= 4\pi \left({r_2}^{2-\alpha} - {r_1}^{2-\alpha} \right) \end{aligned}$$

We have thus verified the divergence theorem as both sides of the equation give the same result.

2.6.2 Stokes' Theorem

Let S be the oriented surface with a boundary curve ∂S. then the integral of $\underline{F}$ around the boundary is given by:

$$\oint_{\partial S} \underline{F} \cdot d\underline{l} = \int_S \nabla \times \underline{F} \cdot d\underline{S} \tag{2.23}$$

This allows us to turn a complicated line integral into a simple surface integral.

Use Stokes' Theorem to evaluate

$$I = \oint_C \underline{A} \cdot d\underline{r}$$

where the square closed contour C has vertices at $(-1,-1,0),(1,-1,0),(1,1,0),(-1,1,0)$ *and* $\underline{A}=(yz-y,\ xz+x,\ xy-z)$.

Let us first compute $\nabla\times\underline{A}$:

$$\begin{aligned}\nabla\times\underline{A}&=\begin{vmatrix}\underline{e}_1 & \underline{e}_2 & \underline{e}_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz-y & xz+x & xy-z\end{vmatrix}\\&=(0,0,2)\end{aligned}$$

As the region lies entirely inside the xy plane, this makes the resultant integral very easy to compute. Using Stokes' Theorem:

$$\begin{aligned}\oint_C \underline{A}\cdot d\underline{r}&=\int(\nabla\times\underline{A})\cdot d\underline{S}\\&=\int(0,0,2)\cdot(0,0,1)\,dS\\&=2\int dS\\&=2\int_{-1}^{1}dx\int_{-1}^{1}dy\\&=8\end{aligned}$$

2.6.3 Green's Theorem in a plane

The two-dimensional case of Green's Theorem states that:

$$\oint_{\partial S}Mdx+Ndy=\int_S\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy \tag{2.24}$$

where ∂S is the boundary of S.

Recall the result of (1.24). Greens' Theorem gives us an interesting result when we apply this condition to the integrand of the left-hand side. Suppose that $Mdx+Ndy$ is exact. Then:

$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}$$

This means that:

$$\oint_{\partial S}Mdx+Ndy=0$$

Thus, for an exact differential, *the closed integral around a loop is zero.* We will cover this is more detail in the next section.

2.7 Line Integrals and Conservative Vector Fields

The reason that these two have been grouped together is because they are closely related to one another.

2.7.1 Line Integrals

Line integrals allow us to compute the sum of the changes in the value of some vector field $\underline{F}$ as it moves along some curve or path ∂S. They are generally of two forms:

$$\begin{aligned} I_1 &= \int_{\partial S} \underline{F} \cdot d\underline{l} \\ &= \int_{\partial S} F_x dx + F_y dy + F_z dz \end{aligned}$$

or

$$\begin{aligned} I_2 &= \int_{\partial S} \underline{F} \cdot dl \\ &= \underline{\widehat{x}} \int_{\partial S} F_x dx + \underline{\widehat{y}} \int_{\partial S} F_y dy + \underline{\widehat{z}} \int_{\partial S} F_z dz \end{aligned}$$

These generate a scalar and vector quantities respectively. In general, it helps to parametrise both $\underline{F}$ and $d\underline{l}$ in terms of a single variable that we can evaluate the integral over. For example, if we want to evaluate over a circle of radius a in the xy plane:

$$\begin{aligned} d\underline{l} &= (dx, dy, dz) \\ \underline{r}(\theta, z) &= (a\cos\theta, a\sin\theta, z) \\ d\underline{r} &= ad\theta\,(-\sin\theta, \cos\theta, 0) \end{aligned}$$

We can then perform the integration for $0 < \theta < 2\pi$ when we have evaluated the dot product $\underline{F} \cdot d\underline{l}$.

2.7.2 Conservative Vector Fields

In general, the result of a path integral between A and B depends on the path of the integration. However, there is a special case of vector fields for which this is not the case. They are known as *conservative* vector fields. We have actually already encountered a few of these, such as Newton's Law of Universal Gravitation or Coulomb's Law. Conservative fields have the property that:

$$\oint_{\partial S} \underline{F} \cdot d\underline{l} = 0 \tag{2.25}$$

i.e the integral of a conservative field around a closed loop is always zero. Using Stokes' Theorem:

$$\begin{aligned} \oint_{\partial S} \underline{F} \cdot d\underline{l} &= \int_S (\nabla \times \underline{F}) \cdot d\underline{S} \\ \nabla \times \underline{F} &= 0 \end{aligned}$$

This means that $\underline{\mathrm{F}}$ can be written as the gradient of some exact differential in order to satisfy this result.

$$d\phi = \frac{\partial F_x}{\partial x} dx + \frac{\partial F_y}{\partial y} dy + \frac{\partial F_z}{\partial z} dz$$

$$\rightarrow \underline{F} = \nabla\phi \tag{2.26}$$

This leads to another result concerning conservative fields:

$$\begin{aligned}\nabla\phi &= \underline{F} \\ \nabla\phi \cdot d\underline{l} &= \underline{F} \cdot d\underline{l} \\ &\approx \int \underline{F} \cdot d\underline{l}\end{aligned}$$

for a small change $d\underline{l}$.

$$\begin{aligned}\nabla\phi \cdot d\underline{l} &\approx d\phi \\ &= \phi(\underline{b}) - \phi(\underline{a})\end{aligned}$$

Hence we arrive at the final result of:

$$\int_a^b \underline{F} \cdot d\underline{l} = \phi(\underline{b}) - \phi(\underline{a}) \tag{2.27}$$

for a conservative field $\underline{F}$. This is actually a very useful result, as it means that in order to compute a line integral of a conservative field, we just have to find the scalar function ϕ.

Compute the line integral

$$I = \int \underline{F} \cdot d\underline{r}$$

from $\underline{a} = (0,0,0)$ *to* $\underline{b} = (1,1,2)$ *along some arbitrary curve, where* $\underline{F} = (y^3, 3xy^2 + yz^2, zy^2)$.

It is clear that $\nabla \times \underline{F} = 0$. We now need to find the corresponding ϕ. We can do this by integrating each component of $\underline{F}$ with respect to the corresponding variable, and then finding the final result by inspection.

$$\begin{aligned}\phi_x &= \int y^3 dx \\ &= y^3 x + C_1(z,y) \\ \phi_y &= \int 3xy^2 + yz^2 \, dy \\ &= xy^3 + \frac{1}{2}y^2z^2 + C_2(x,z) \\ \phi_z &= \int zy^2 dz \\ &= \frac{1}{2}z^2y^2 + C(x,y)\end{aligned}$$

The constants have been included to remind the reader that we are only doing integration with respect to one of the variables; there may be constants that are functions of other variables. Consequently:

$$\begin{aligned}\phi &= xy^3 + \frac{1}{2}z^2y^2 \\ \rightarrow I &= 3\end{aligned}$$

3. *Normal Modes and Waves*

This chapter focusses on the basics of normal mode and wave theory, including:

- Normal Mode methods
- The Wave Equation
- Waves on a string
- Waves at Boundaries
- Dispersion

This section essentially deals with harmonic motion of many kinds, and generally it is assumed that energy is conserved in these systems; we do not take account of damping. Furthermore, the types of questions covered in this section are highly dependant on the consideration of boundary conditions; always be thinking about how the problem might be bounded or constrained when attempting such questions.

3.1 Normal Mode Methods

A *normal mode* is a characteristic state of oscillation of a system around some state of equilibrium. This kind of behaviour is encountered in a large variety of physical systems, and the methods outlined in this chapter allow quite complicated systems to be characterised solely by a set of normal mode frequencies. In general, normal modes represent some sort of analogue of either constant motion of the centre of mass of the system, or motion of the masses relative to the centre of mass of the system. Note that after sufficient practise, solving these types of questions becomes quite formulaic as generally the same process is followed each time.

In most cases, one has to solve for the equations of motion of the components of the system by resolving forces using Newton's Second Law. In some rarer scenario's, it is more helpful to consider the kinetic and potential energies of the system. Once these equations of motion have been obtained, we can use two main methods to solve the problem.

3.1.1 Matrix Method

After obtaining the equations of motion, re-arrange the equations into a system of linear equations, and substitute:

$$x_1 = A_1 e^{i\omega t} \text{ and } x_2 = A_2 e^{i\omega t} \tag{3.1}$$

for displacements x_1 and x_2 from equilibrium. This general method for solving coupled equations is outlined in the notes on CP3. We then take the determinant of the coefficient matrix and set it to zero to obtain a valid solution to the homogeneous system. This will give the normal modes of oscillation.

The general solutions are:

$$x_1 = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \tag{3.2}$$

$$x_2 = A_1' \cos(\omega_1 t + \phi_1) + A_2' \cos(\omega_2 t + \phi_2) \tag{3.3}$$

where ω_1 and ω_2 are the normal mode frequencies, and A_1, A_2, ϕ_1 and ϕ_2 are constants to be determined from initial conditions. Note that we find A_1' and A_2' from the amplitude ratios that we obtain from the equations of motion.

Let us consider an example. *Two masses of equal mass m are placed at the same distance l from either end of a string of total length $4l$ and negligible mass under a tension T. One end E is driven harmonically at $h(t) = h_o \cos \omega t$. What are the normal mode frequencies of oscillation? Express your answer in terms of $w_o = \sqrt{T/2ml}$. Find the general solutions for the masses. What happens when $\omega = \sqrt{3}\omega_o$?*

As you can see, these questions are often very wordy to begin with, but after that the methodology is pretty straight forward. Let us first consider a diagram of the situation. By Newton's Second Law, let us equate vertical forces. We shall assume that the displacements are transverse, and that θ_1, θ_2 and θ_3 are sufficiently small such that $\sin\theta \approx \tan\theta$.

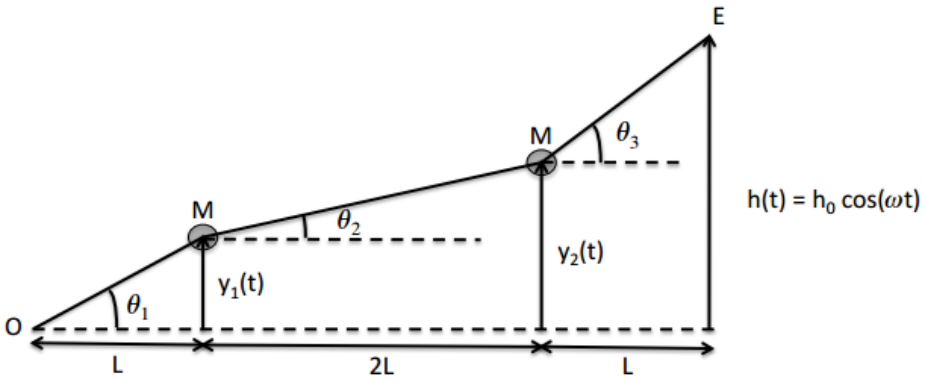

Figure 3.1: Masses on a driven string

For y_1:

$$\begin{aligned} m\ddot{y}_1 &= T(\sin\theta_2 - \sin\theta_1) \\ &\approx T(\tan\theta_2 - \tan\theta_1) \\ \ddot{y}_1 &= \frac{T}{m}\left(\frac{y_2 - y_1}{2l} - \frac{y_1}{l}\right) \\ &= {\omega_o}^2(y_2 - 3y_1) \end{aligned}$$

For y_2:

$$\begin{aligned} m\ddot{y}_2 &= T(\sin\theta_3 - \sin\theta_2) \\ &\approx T(\tan\theta_3 - \tan\theta_2) \\ \ddot{y}_2 &= \frac{T}{m}\left(\frac{h - y_2}{l} - \frac{y_2 - y_1}{2l}\right) \\ &= {\omega_o}^2(3h_o\cos\omega t - 3y_2 + y_1) \end{aligned}$$

Writing the equations as a system of linear equations:

$$\begin{aligned} \ddot{y}_1 + 3{\omega_o}^2 y_1 - {\omega_o}^2 y_2 &= 0 \\ -{\omega_o}^2 y_1 + \ddot{y}_2 + 3{\omega_o}^2 y^2 &= 2{\omega_o}^2\, h_o\cos\omega t \end{aligned}$$

Making the substitutions outlined in (3.1), we arrive at the matrix equation:

$$\begin{pmatrix} -\omega^2 + 3{\omega_o}^2 & -{\omega_o}^2 \\ -{\omega_o}^2 & -\omega^2 + 3{\omega_o}^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2{\omega_o}^2 \end{pmatrix}$$

We require that the determinant of the coefficient matrix is zero for a solution to this system to exist.

$$\begin{aligned} (-\omega^2 + 3{\omega_o}^2)^2 &= {\omega_o}^4 \\ -\omega^2 + 3{\omega_o}^2 &= \pm{\omega_o}^2 \\ w^2 &= 3{\omega_o}^2 \pm {\omega_o}^2 \end{aligned}$$

Hence:

$$\begin{aligned} \omega_1 &= \sqrt{2}\omega_o \\ \omega_2 &= 2\omega_o \end{aligned}$$

A check that can always be done at this point to check whether these frequencies are positive, as if they are not, then something has gone horribly wrong.

Now we have to find the amplitude ratios. If we rearrange the first of the equations in the matrix system:

$$\begin{aligned}(-\omega^2 + 3{\omega_o}^2)A_1 - {\omega_o}^2 A_2 &= 0 \\ \frac{A_2}{A_1} &= \frac{1}{{\omega_o}^2}(-\omega^2 + 3{\omega_o}^2) \\ \left(\frac{A_2}{A_1}\right)_1 &= \frac{1}{{\omega_o}^2}(-2{\omega_o}^2 + 3{\omega_o}^2) \\ &= 1 \\ \left(\frac{A_2}{A_1}\right)_2 &= \frac{1}{{\omega_o}^2}(-4{\omega_o}^2 + 3{\omega_o}^2) \\ &= -1\end{aligned}$$

Thus, we can immediately write down the most general solutions for the homogeneous motion of the two masses:

$$\begin{aligned}y_1 &= C_1\cos(\omega_1 t + \phi_1) + C_2\cos(\omega_2 t + \phi_2) \\ y_2 &= C_1\cos(\omega_1 t + \phi_1) - C_2\cos(\omega_2 t + \phi_2)\end{aligned}$$

Let us now consider the steady-state motion of the two masses. For this, we need to solve the inhomogeneous part of the matrix system. Use the trial substitutions of $y_1 = c_1\cos\omega t$ and $y_2 = c_2\cos\omega t$ in the two equations for the system. Consequently:

$$\begin{aligned}c_1(3{\omega_o}^2 - \omega^2) &= {\omega_o}^2 c_2 \\ c_2(3{\omega_o}^2 - \omega^2) &= {\omega_o}^2 c_1 + 2{\omega_o}^2 h_o \\ c_2 &= \frac{2h_o{\omega_o}^4(3{\omega_o}^2 - \omega^2)}{(2{\omega_o}^2 - \omega^2)(4{\omega_o}^2 - \omega^2)} \\ c_1 &= \frac{2{\omega_o}^4 h_o}{(2{\omega_o}^2 - \omega^2)(4{\omega_o}^2 - \omega^2)}\end{aligned}$$

Lastly, we are asked *What happens when* $\omega = \sqrt{3}\omega_o$ *?*

$$\begin{aligned}y_1 &= \frac{2{\omega_o}^4 h_o}{(-{\omega_o}^2)({\omega_o})^2}\cos\omega t \\ &= -2h_o\cos\omega t \\ &= 2h_o\cos(\omega t + \pi) \\ y_2 &= 0\end{aligned}$$

At this frequency, the second mass behaves like a fixed node while the first performs oscillations at twice the amplitude but π out of phase with the driving force $h(t)$.

3.1.2 Decoupling Method

This method is vastly less flexible than the matrix method (and is consequently less frequently used) but can be a bit of a short-cut in some questions. The *decoupling method* involves taking the sum, difference or other linear combination of the equations of motion to obtain ordinary differential equations in the *normal mode coordinates* q_1 and q_2. We can then find the solutions to the displacements x_1 and x_2 by taking the reverse linear combination of the solved normal mode coordinates. For example:

$$q_1 = (x_1 + x_2) \text{ and } q_2 = (x_1 - x_2) \tag{3.4}$$

Let us again consider an example. *Two identical pendula consist of a light string of length l attached to a mass m. They are joined by a spring of spring constant k. Find the normal mode frequencies of the system.*

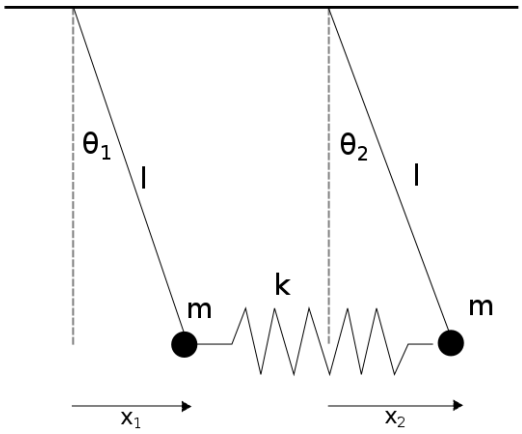

Figure 3.2: Two identical pendula joined by a spring

We have to take into account the small-angle gravitational restoring force and the Hookean restoring force due to the spring.

$$\begin{aligned} m\ddot{x}_2 &= -x_2 \sin\theta_2 - k(x_2 - x_1) \\ &\approx -x_2 \tan\theta_2 - k(x_2 - x_1) \\ &= -\frac{mg}{l}\, x_2 + k(x_2 - x_1) \\ m\ddot{x}_1 &= -\frac{mg}{l} x_1 + k(x_2 - x_1) \end{aligned}$$

Taking the sum of these two equations:

$$\begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) &= -\frac{mg}{l}(x_1 + x_2) \\ \ddot{q}_1 + \frac{g}{l} q_1 &= 0 \\ \rightarrow \omega_1 &= \sqrt{\frac{g}{l}} \end{aligned}$$

Taking the difference of these two equations:

$$\begin{aligned} m(\ddot{x}_1 - \ddot{x}_2) &= -\left(\frac{mg}{l} + 2k\right)(x_1 - x_2) \\ \ddot{q}_2 + (\frac{g}{l} + \frac{2k}{m}) q_2 &= 0 \\ \rightarrow \omega_2 &= \sqrt{\frac{g}{l} + \frac{2k}{m}} \end{aligned}$$

As you can see, in this case the decoupling method was much faster than the matrix method, but as you can imagine, this only works for relatively simple systems.

3.2 The Wave Equation

The rest of the material in this chapter essentially deals with solutions to a specific case of the general wave equation related to waves on a string. In this section, we deal with it's derivation and solution.

3.2.1 Derivation

Consider the small section of string shown in the diagram below:

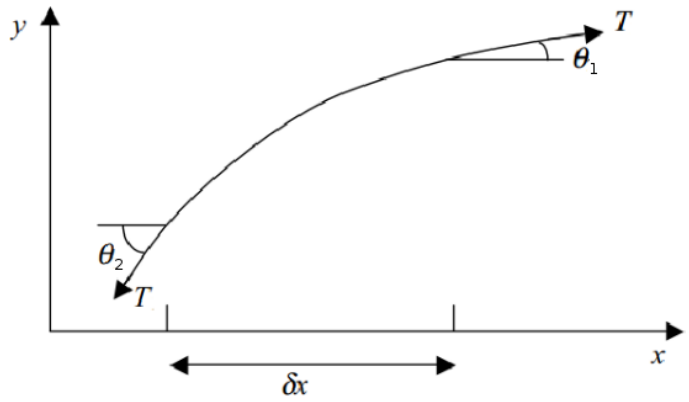

Figure 3.3: A small section of stretched string

Begin by resolving forces vertically by Newton's Second Law and using the small angle approximation $\sin\theta \approx \tan\theta$ for small angles θ.

$$\begin{aligned} T[\sin\theta_1 - \sin\theta_2] &= m\frac{\partial^2 y}{\partial t^2} \\ T[\tan\theta_1 - \tan\theta_2] &\approx (\rho\delta x)\frac{\partial^2 y}{\partial t^2} \\ &= T\left[\left(\frac{\partial y}{\partial x}\right)_1 - \left(\frac{\partial y}{\partial x}\right)_2\right] \end{aligned}$$

Expanding $\left(\frac{\partial y}{\partial x}\right)_1$ by the Taylor Series and keeping first order terms:

$$\begin{aligned} \left(\frac{\partial y}{\partial x}\right)_1 &= \left(\frac{\partial y}{\partial x}\right)_2 + \delta x\,\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) \\ T\left[\left(\frac{\partial y}{\partial x}\right)_2 + \delta x\,\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) - \left(\frac{\partial y}{\partial x}\right)_2\right] &= (\rho\delta x)\frac{\partial^2 y}{\partial t^2} \\ (T\delta x)\frac{\partial^2 y}{\partial x^2} &= (\rho\delta x)\frac{\partial^2 y}{\partial t^2} \end{aligned}$$

Thus, we obtain the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\,\frac{\partial^2 y}{\partial t^2} \tag{3.5}$$

The waves on the string thus have speed $c = \sqrt{T/\rho} = \omega/k$.

3.2.2 D'Alembert's Solution

We know that the solution to the wave equation must be a function of displacement and time; that $y = y(x,t)$. Let us define new variables u and v such that y can also be written as a function of u and v.

$$u = x - ct$$
$$v = x + ct$$

Using Leibnitz's Theorem (1.3):

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\,\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$
$$\frac{\partial^2 y}{\partial t^2} = c^2\left[\frac{\partial^2 y}{\partial u^2} - 2\,\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right]$$

Substituting these results into the wave equation forces $y_{uv} = 0$. This implies that in the total derivative expression

$$dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv$$

$\frac{\partial y}{\partial u}$ is a function only of u, and $\frac{\partial y}{\partial v}$ is a function only of v. We can thus write:

$$\begin{aligned} y &= \int \frac{\partial y}{\partial u}\,du + \int \frac{\partial y}{\partial v}\,dv \\ &= f(u) + g(v) \\ \rightarrow y &= f(x-ct) + g(x+ct) \end{aligned}$$

Suppose that at $t = 0$ the initial displacement of the string is given by $u(x)$ and the initial velocity by $v(x)$.

$$u(x) = f(x) + g(x)$$
$$v(x) = -cf'(x) + cg'(x)$$

Integrating the second equation, and then solving for $f(x)$ and $g(x)$:

$$\begin{aligned} f(x) - g(x) &= \frac{1}{c}\int_b^x v(x)\,dx \\ g(x) &= \frac{1}{2}u(x) + \frac{1}{2c}\int_b^x v(x)dx \\ f(x) &= \frac{1}{2}u(x) - \frac{1}{2c}\int_b^x v(x)dx \end{aligned}$$

Thus, the general solution for all time, given d'Alembert's method, is:

$$y(x,t) = \frac{1}{2}\left[u(x-ct) + u(x+ct)\right] + \frac{1}{2c}\left[\int_b^{x+ct} v(x)dx - \int_b^{x-ct} v(x)dx\right]$$

$$y(x,t) = \frac{1}{2}\left[u(x-ct) + u(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(x)\,dx \tag{3.6}$$

Thus, given the initial conditions of the displacement and velocity of the string, we can find the subsequent motion of the string for all time assuming that no energy is lost from the system.

3.2.3 Separation of Variables

In this method, we assume that the solution can be written as the product of two functions that are dependant only on time and displacement respectively:

$$y(x,t) = g(x)\, f(t)$$

Substitute this in to the wave equation:

$$f(t)\, g''(x) = \frac{1}{c^2}\, g(x)\, \ddot{f}(t)$$
$$\frac{g''(x)}{g(x)} = \frac{1}{c^2}\, \frac{\ddot{f}(t)}{f(t)}$$

This means that both sides must be equal to some separation constant, which for convenience's sake we will call $-k^2$. We thus obtain the following ordinary differential equations:

$$g''(x) + k^2\, g(x) = 0$$
$$\ddot{f}(t) + c^2 k^2\, f(t) = 0$$

In general, the separated parts will have solutions:

$$g(x) = A\cos kx + B\sin kx \tag{3.7}$$
$$f(t) = C\cos kct + D\sin kct \tag{3.8}$$

Initial conditions can then be imposed on these solutions to find the subsequent displacement of the string.

3.3 Waves on a String

Given that we have derived the wave equation for waves on a string, there are some properties and cases that we can derive alongside this result.

3.3.1 Energy of Oscillation

Consider again the infinitesimally small section of string shown in Figure (3.3). Let the length along the string between x and $x + dx$ be ds. First, let us derive an expression for the kinetic energy:

$$\begin{aligned} KE &= \frac{1}{2}(\rho dx)\,\dot{y}^2 \\ &= \frac{1}{2}\rho dx \left(\frac{\partial y}{\partial t}\right)^2 \\ \rightarrow KE &= \int \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 dx \end{aligned} \tag{3.9}$$

Now for the potential energy. This is equal to the work done to stretch the string.

$$\begin{aligned} U &= T(ds - dx) \\ ds &= \sqrt{dx^2 + dy^2} \\ &= dx\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \\ &\approx dx\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right) \\ \rightarrow U &= \int \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx \end{aligned} \tag{3.10}$$

Note that these two expressions are of equal magnitude for a wave that obeys the wave equation as $Tk^2 = \rho\omega^2$.

The power of the wave is the flow of energy per unit time down the string.

$$\begin{aligned} Power &= \frac{Energy}{Wavelength} \times \frac{Distance\ Travelled}{Time} \\ P &= \frac{1}{2}\rho A^2\omega^2 c \end{aligned}$$

where A is the amplitude of the wave.

3.3.2 Standing Waves on a String

A standing wave on a string is formed from the superposition of two waves of equal amplitude travelling in opposite directions:

$$\begin{aligned} y(x,t) &= A_o\cos(\omega t + kx) + A_o\cos(\omega t - kx) \\ &= 2A_o\sin kx\cos\omega t \end{aligned}$$

This can only occur when both ends of the string are fixed. Suppose that the string has length L. We need to impose the conditions that $y(0,t) = 0$ and $y(L,t) = 0$.

$$\begin{aligned} \sin kL &= 0 \\ kL &= n\pi \\ k &= \frac{n\pi}{L} \end{aligned}$$

This means that the frequencies of the normal mode standing waves are given by:

$$\omega_n = \frac{cn\pi}{L} \tag{3.11}$$

for integer $n \neq 0$. This condition can also be derived from the solutions in (3.8) by imposing the condition that the string is initially at rest ($\partial y/\partial t = 0$) before dealing with the condition on the ends.

Given some initial displacement of the string, one finds the subsequent evolution of the string by simply multiplying by the corresponding time-varying amplitude for each of the normal modes involved.

The initial velocity of a string fixed at $x = 0$ and $x = L$ is given by

$$\frac{\partial y}{\partial t}(x,0) = v_o \, \sin^5\left(\frac{\pi x}{L}\right)$$

If at $t = 0$ it is initially in it's equilibrium position, find the subsequent displacement of the string.

We need to decompose this initial velocity into a series of normal mode velocities, as then the remainder of the problem becomes relatively straight forward. We can do this quite easily with complex numbers. Recall that:

$$\begin{aligned} \sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \\ \sin^5\theta &= \frac{1}{2^4}\cdot\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)^5 \\ &= \frac{1}{2^4}\cdot\frac{1}{2i}\left(\left(e^{5i\theta} - e^{-5i\theta}\right) - 5\left(e^{3i\theta} - e^{-3i\theta}\right) + 10\left(e^{i\theta} - e^{-i\theta}\right)\right) \\ &= \frac{1}{2^4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta) \end{aligned}$$

We have thus found that the initial velocity of the string involves components corresponding to the $n = 1$, $n = 3$ and $n = 5$ normal modes.

$$\begin{aligned} \frac{\partial y}{\partial t}(x,0) &= v_o \, \sin^5\left(\frac{\pi x}{L}\right) \\ &= \frac{v_o}{16}\left(\sin\left(\frac{5\pi x}{L}\right) - 5\sin\left(\frac{3\pi x}{L}\right) + 10\sin\left(\frac{\pi x}{L}\right)\right) \\ \frac{\partial y}{\partial t}(x,t) &= \frac{v_o}{16}\left(\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi ct}{L}\right) - 5\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right) + 10\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right)\right) \\ y(x,t) &= \frac{v_o\, L}{80\pi c}\sin\left(\frac{5\pi x}{L}\right)\sin\left(\frac{5\pi ct}{L}\right) - \frac{5v_o\, L}{48\pi c}\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{3\pi ct}{L}\right) + \frac{5v_o\, L}{8\pi c}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi ct}{L}\right) \end{aligned}$$

This is the final solution for the subsequent displacement of the string.

Let us now examine the energy of the string. We have already found that a standing wave on a fixed length L is of the form:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \tag{3.12}$$

If we substitute (3.12) into (3.9) and (3.10), we find that:

$$KE_n = \frac{\rho\, {A_n}^2\, (n\pi c)^2}{4L} \sin^2\left(\frac{n\pi ct}{L}\right)$$
$$U_n = \frac{T\, {A_n}^2\, (n\pi)^2}{4L} \cos^2\left(\frac{n\pi ct}{L}\right)$$

Adding these two together, it becomes clear that the energy for each normal mode is constant in time, and equal to:

$$E_n = \frac{\rho\, c^2\, (A_n \pi n)^2}{4L} \tag{3.13}$$

The energy of the string, when more than one normal mode is present, is just the sum of the energies in all the normal modes. This should be conserved, assuming that the string is isolated and not damped.

3.4 Waves at Boundaries

When a travelling wave encounters some sort of boundary (whether on a string or otherwise), the boundary has some effect on the propagation of said wave. It may reflect some of the energy, reduce the amplitude of the wave, or have a variety of other effects.

3.4.1 Boundaries on a string

Below are a variety of examples of some of the boundaries that a wave might be able to encounter when it is constrained to travel on a string. We shall assume that the frequency of the waves is unchanged at the boundaries, as well as the fact that energy is conserved. Note that we will not go through the derivations for all the results (just the first one) as the method very similar for all of them.

- Density Boundary - Suppose that two strings of different densities are attached at $x = 0$. In this case, we require that the string remains continuous; this means that each string section must have the same displacement, and gradient at the join. This results in the boundary conditions of

$$y\Big|_{x=0^-} = y\Big|_{x=0^+} \tag{3.14}$$

$$\frac{\partial y}{\partial x}\Big|_{x=0^-} = \frac{\partial y}{\partial x}\Big|_{x=0^+} \tag{3.15}$$

We can then use these conditions to find the behaviour of the wave at the join.

Suppose that a travelling wave of amplitude A_1 travels from $x = -\infty$ along a semi-infinite string of density ρ_1. At $x = 0$ it encounters a second semi-infinite string of density ρ_2. Find the amplitudes of the transmitted and reflected waves.

To solve this problem, we need to solve for A_2 and A_3 in the following three waves:

1. Incident: $y_1 = A_1\, e^{i(\omega t - k_1 x)}$
2. Reflected: $y_2 = A_2\, e^{i(\omega t + k_1 x)}$
3. Transmitted: $y_3 = A_3\, e^{i(\omega t - k_2 x)}$

We have included different wave-numbers for the waves travelling in different densities as wave-number is related to density by $k = c^{-1}\sqrt{\rho/T}$. Note that the exponents of (1) and (2) differ solely by a sign; this is because the reflected wave travels towards $-\infty$ while the incident wave is originally travelling towards $+\infty$.

Substituting these waves into (3.15), where y_1 and y_2 correspond to $x = 0^-$ and y_3 corresponds to $x = 0^+$, we obtain (after some cancellation):

$$A_1 + A_2 = A_3$$
$$-k_1 A_1 + k_1 A_2 = -k_2 A_3$$

Solving these two equations simultaneously, we obtain the amplitudes of the reflected and transmitted waves as:

$$\frac{A_{reflected}}{A_{incident}} = \frac{k_1 - k_2}{k_1 + k_2} \tag{3.16}$$

$$\frac{A_{transmitted}}{A_{incident}} = \frac{2k_1}{k_1 + k_2} \tag{3.17}$$

Now it is worth considering a couple of limiting cases.

- $\rho_2 = 0$ - This essentially corresponds to not having a second string; this means that all the energy must be reflected at the boundary, and so the amplitude and phase of the reflected wave are the same.

$$A_{transmitted} \to 0$$
$$A_{reflected} \to A_{incident}$$

- $\rho_2 = \rho_1$ - This means that there is no boundary to disturb the wave, and so it just passes through as expected with no phase or amplitude change.

$$A_{transmitted} \to A_{incident}$$
$$A_{reflected} \to 0$$

- $\rho_2 \to \infty$ - This is the "brick wall" scenario. As no energy can pass onto the second string, it is all reflected, but in this case, there is a phase change of π.

$$A_{transmitted} \to 0$$
$$A_{reflected} \to -A_{incident}$$

- Terminated by a mass - In this case, we need to equate the force on the mass due to the incident wave and the acceleration it feels by Newton's Second Law. This results in the condition that:

$$T \frac{\partial y}{\partial x}\Big|_{x=0^-} = m\frac{\partial^2 y}{\partial t^2} \tag{3.18}$$

For an incident wave travelling on a semi-infinite string from $x = -\infty$ towards $x = 0$, this yields:

$$\frac{A_{reflected}}{A_{incident}} = e^{i\phi} \tag{3.19}$$
$$\phi = 2\ \tan^{-1}\left(\frac{kT}{m\omega^2}\right) \tag{3.20}$$

Evidently, there are no transmitted waves as the string ends with the mass. This means all the energy is reflected, but a complex phase ϕ is introduced. Note the limiting cases:

- $m \to \infty$, $\phi \to 0$
- $m \to 0$, $\phi \to \pi$

These agree with the limiting cases discussed above.

- 'Forced' Mass Boundary - Suppose that we have a mass attached to two strings of the same density at $x = 0$ that is itself attached to a spring of spring constant s. We have opted not to use the conventional k here because we are using it to refer to wave-number. Once again, we need to equate forces at the boundary:

$$T\left(\frac{\partial y}{\partial x}\Big|_{x=0^-} - \frac{\partial y}{\partial x}\Big|_{x=0^+}\right) = m\frac{\partial^2 y}{\partial t^2} + sy \tag{3.21}$$

Again, *for an incident wave travelling on a semi-infinite string from $x = -\infty$ towards $x = 0$*, this yields:

$$\frac{A_{reflected}}{A_{incident}} = \frac{(k_1 - k_2)T - i(m\omega^2 - s)}{(k_1 + k_2)T + i(m\omega^2 - s)} \tag{3.22}$$
$$\frac{A_{transmitted}}{A_{incident}} = \frac{2Tk_1}{(k_1 + k_2)T + i(m\omega^2 - s)} \tag{3.23}$$

For the case where $s = 0$, we obtain the case if we just had a mass attached at the boundary of the two strings.

- 'Triple Boundary' - This is more of an excuse to include a question rather than be particularly illustrative. The following has mostly been left as an exercise for the reader, with a few hints thrown in. Be warned; this is quite an algebraically heavy question.

  A wave of unit amplitude travels towards $x = +\infty$ from $x = -\infty$ along a semi-infinite string of density ρ_1. At $x = 0$ it encounters a second string of density ρ_2 that is itself attached to another string of density ρ_3 at $x = a$. For $a = \lambda_2$ (where λ_2 is the wavelength of the waves in the middle section), show that there is no reflection at the initial boundary, and find the amplitude of the wave transmitted to $x = +\infty$.

  The starting point when tackling this question should be to impose the conditions shown in (3.15) both at $x = 0$ and $x = a$, requiring us to solve for 5 amplitudes (chugging through the simultaneous equations is where the maths gets heavy). We can also use the condition that $a = \lambda_2$ to simplify this a little as:

$$e^{\pm ik_2 a} = e^{\pm i\frac{2\pi}{\lambda_2}\lambda_2} = e^{\pm i\,2\pi} = 1$$

  The answer for the amplitude turns out to be:

$$A_{\,Final} = \frac{2k_1}{k_1 + k_3}$$

  It is as if the wave 'skips' the middle section.

3.4.2 Boundaries in Circuits

What? There are boundaries in circuits? The response is evidently yes, and this is because we can model 'current' and 'voltage' waves within circuits that can encounter an impedance or resistance. First, let us derive the relevant wave equations.

Consider a circuit of capacitance per unit length C and inductance per unit length L. The voltage drop due to the inductance over a small distance δx is given by Faraday's Law:

$$\begin{aligned} \delta V &= -(L\delta x)\,\frac{\partial I}{\partial t} \\ \rightarrow \frac{\partial V}{\partial x} &= -L\frac{\partial I}{\partial t} \end{aligned}$$

The voltage drop due to capacitance over δx is given by:

$$\begin{aligned} \delta V &= -\frac{1}{C}\,\frac{\partial Q}{\partial x} \\ \rightarrow C\frac{\partial V}{\partial t} &= -\frac{\partial I}{\partial x} \end{aligned}$$

These are known as the telegraph equations. If we differentiate each equation appropriately and substitute into one another, we arrive at:

$$\frac{\partial^2 V}{\partial x^2} = LC\,\frac{\partial^2 V}{\partial t^2} \tag{3.24}$$

$$\frac{\partial^2 I}{\partial x^2} = LC\,\frac{\partial^2 I}{\partial t^2} \tag{3.25}$$

where $c = 1/\sqrt{LC}$.

The characteristic impedance for a circuit is defined as:

$$Z \equiv \frac{V_o}{I_o} = \pm\sqrt{\frac{L}{C}} \tag{3.26}$$

Suppose that these waves encounter an impedance boundary of terminating impedance z_T. Here, the ratio of V/I must equal the terminating impedance. Let the incident amplitude be A and the reflected amplitude be A'.

$$\begin{aligned} V &= Ae^{i(\omega t - kx)} + A'e^{i(\omega t + kx)} \\ Z_o I &= Ae^{i(\omega t - kx)} - A'e^{i(\omega t + kx)} \\ Z_T &= \frac{V(0,t)}{I(0,t)} \\ \frac{Z_T}{Z_o} &= \frac{A + A'}{A - A'} \end{aligned}$$

This means that the reflected wave amplitude is given by:

$$\frac{A_{\,Reflected}}{A_{\,Incident}} = \frac{Z_T - Z_o}{Z_T + Z_o} \tag{3.27}$$

Let us consider some limiting cases:

- $Z_T = 0$ - All the energy is reflected with a phase shift of π.

$$A_{\,reflected} \to -A_{\,incident}$$

- $Z_T = Z_o$ - No energy is reflected. This is called matched impedance, where all the power is transmitted to the terminating load.

$$A_{\,reflected} \to 0$$

- $Z_T \to \infty$ - All the energy is reflected but with no phase shift.

$$A_{\,reflected} \to A_{\,incident}$$

3.5 Dispersion

A *dispersive medium* is on in which different frequencies are transmitted at different speeds. This is responsible for effects such as the dispersion of light by prisms, as different wavelengths are diffracted by different amounts. We have two characteristic speeds associated with waves in dispersive media:

3.5.1 Phase Velocity (v_p)

This is the speed of an individual wavelength in a medium.

$$v_p = \frac{\omega}{k} \tag{3.28}$$

This may actually be greater than the speed of light, but this does not violate causality as the information of a wave is actually transmitted with the *Group Velocity* that is always less than the speed of light.

3.5.2 Group Velocity (v_g)

This is the speed at which the 'envelope' compromising of different frequencies travels, though the envelope may vary with time. Consider two waves y_1 and y_2 that have the same amplitude but differ in frequency and wave-number by a small amount:

$$\begin{aligned} y_1 &= A\sin((k+\delta k)x - (\omega+\delta\omega)t) \\ y_2 &= A\sin((k-\delta k)x - (\omega-\delta\omega)t) \end{aligned}$$

Finding the resultant wave from the superposition of the two waves:

$$\begin{aligned} y &= y_1 + y_2 \\ &= 2A\cos(\delta k x - \delta\omega t)\sin(kx - \omega t) \end{aligned}$$

Thus, the wave packet moves at $c = \delta\omega/\delta k$ for the superposition of two waves. For a large number of waves, this becomes:

$$v_g = \frac{dw}{dk} \tag{3.29}$$

We can prove this more generally by considering the superposition of waves using an integral:

$$\begin{aligned} y(x,t) &\propto \int y_k\, e^{i(kx-\omega(k)t)}\, dk \\ &= \int y_k\, e^{if(k)}\, dk \end{aligned}$$

The wave packet will be stable for $\partial f(k)/\partial k$.

$$\begin{aligned} \frac{\partial f}{\partial k} &= 0 \\ x - \frac{\partial w}{\partial t}\, t &= 0 \\ v_p = \frac{x}{t} &= \frac{\partial\omega}{\partial k} \end{aligned}$$

There are some useful expression for v_g that we can find by manipulation of it's original definition.

- In terms of v_p and wave-number k :

$$\begin{aligned} v_g &= \frac{d}{dk}(v_p\, k) \\ \rightarrow v_g &= v_p + k\frac{dv_p}{dk} \end{aligned}$$

- In terms of v_p and wavelength λ :

$$\begin{aligned} k &= \frac{2\pi}{\lambda} \\ dk &= -\frac{2\pi}{\lambda^2}\, d\lambda \\ \rightarrow v_g &= v_p - \lambda\frac{dv_p}{d\lambda} \end{aligned}$$

- In terms of the refractive index n, the speed of light in a vacuum c and wavelength λ :

$$\begin{aligned} v_p &= \frac{c}{n} \\ \frac{dv_p}{dn} &= -\frac{c}{n^2} \\ v_g &= \frac{c}{n} - \lambda\frac{d}{d\lambda}\left(-\frac{c}{n^2}dn\right) \\ &= \frac{c}{n} + \frac{c}{n}\left(\frac{d}{d\lambda}\cdot\frac{dn}{n}\right) \\ \rightarrow v_g &= \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{\partial n}{\partial\lambda}\right) \end{aligned}$$

- In terms of the refractive index n, the speed of light in a vacuum c and frequency ω :

$$\begin{aligned} \frac{c}{n} &= \frac{w}{k} \\ k &= \frac{\omega n(w)}{c} \\ \frac{1}{v_g} &= \frac{d}{dw}(k) \\ &= \frac{d}{dk}\left(\frac{\omega n(w)}{c}\right) \\ \rightarrow \frac{1}{v_g} &= \frac{n}{c}\left(1 + \frac{\omega}{n}\frac{\partial\omega}{\partial n}\right) \end{aligned}$$

The utility of each of these depends on the dispersion relation involved. For all physical media, phase and group velocity should obey the relation:

$$v_g v_p = c^2 \tag{3.30}$$

3.5.3 Local Wave-number and Frequency

Thus far, we have assumed that ω and k are constant throughout space for a given wave. What happens if we introduce local frequency and wave-number? For $y = Ae^{i\theta}$:

$$\begin{aligned} \theta &= \bar{k}x - \bar{\omega}t + \theta_o \\ &= k(x,t)\, x - \omega(x,t)\, t + \theta_o \end{aligned}$$

For this to be solution to the wave equation:

$$\begin{aligned}\frac{\partial\theta}{\partial x} &\stackrel{!}{=} k \\ \frac{\partial\theta}{\partial x} &\approx k + \frac{\partial k}{\partial x}x \\ \frac{\partial\theta}{\partial t} &\stackrel{!}{=} \omega \\ \frac{\partial\theta}{\partial t} &\approx \omega + \frac{\partial\omega}{\partial t}t\end{aligned}$$

This means that we require for both $k(x,t)$ and $\omega(x,t)$ to be slowly, varying:

$$\frac{\partial k}{\partial x} = \frac{\partial\omega}{\partial t} \approx 0$$

Furthermore, we require that

$$d\theta = \bar{k}\,dx - \bar{\omega}\,dt + \theta_o$$

is a total differential. This gives:

$$\frac{\partial\bar{k}}{\partial t} + \frac{\partial\bar{\omega}}{\partial x} = 0$$

Assuming that $\omega = \omega(k)$:

$$\begin{aligned}\frac{\partial\bar{\omega}}{\partial x} &= \frac{\partial\bar{k}}{\partial x}\frac{\partial\bar{\omega}}{\partial x} \\ &= v_g\frac{\partial\bar{k}}{\partial x}\end{aligned}$$

This leads to the wave conservation equations for local wave-number and frequency:

$$\frac{\partial\bar{k}}{\partial t} + v_g\frac{\partial\bar{k}}{\partial x} = 0 \tag{3.31}$$

$$v_g\frac{\partial\bar{\omega}}{\partial t} + \frac{\partial\bar{\omega}}{\partial x} = 0 \tag{3.32}$$

3.5.4 A Dispersion Relation example

This is just an example of a harder dispersion relation question, though it is illustrative of quite a few useful techniques.

Suppose that

$$\phi = a\left[e^{i(\omega t - kx)} + e^{i(\omega' t - k' x)}\right]$$

In a certain dispersive medium ϕ obeys the equation

$$\tau\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial t^2} - {c_1}^2\frac{\partial^2\phi}{\partial x^2}\right) + \frac{\partial^2\phi}{\partial t^2} - {c_o}^2\frac{\partial^2\phi}{\partial x^2} = 0$$

for $c_1 < c_o$. Show that a disturbance with frequency $\omega \ll 1/\tau$ travels with phase velocity c_o, and that it's amplitude decreases by a factor

$$\approx \exp\left[-\pi\omega\tau\left(\frac{{c_1}^2}{{c_o}^2} - 1\right)\right] \tag{3.33}$$

Let $\delta = i(\omega t - kx)$ and $\delta' = i(\omega' t - k' x)$. With careful differentiation and substitution into the dispersion relation, it can be shown that:

$$\tau a i e^{\delta}(\omega {c_1}^2 k^2 - \omega^3) + \tau a i e^{\delta'}(\omega {c_1}^2 k'^2 - \omega'^3) + a e^{\delta}({c_o}^2 k^2 - \omega^2) + a e^{\delta'}({c_o}^2 k'^2 - \omega'^2) = 0$$

Considering the part of the wave with frequency $\omega \ll 1/\tau$:

$$\tau a i e^{\delta}(\omega {c_1}^2 k^2 - \omega^3) + a e^{\delta}({c_o}^2 k^2 - \omega^2) = 0$$

As τ, a and e^{δ} are all non-zero, this means that both the real and imaginary parts must be zero for this relation to hold:

$$\begin{aligned} {c_o}^2 k^2 - \omega^2 &= 0 \\ \omega^2 &= {c_o}^2 k^2 \\ \omega &= \pm c_o k \\ \frac{\omega}{k} &= \pm c_o \\ v_p &= \pm c_o \end{aligned}$$

Hence a wave with frequency ω travels with a phase velocity c_o. Letting $\omega = \omega_o + \delta\omega = \pm c_o k + \delta\omega$ in (3.33).

$$i\tau(\omega_o + \delta\omega)^3 - i\tau k^2 {c_1}^2(\omega_o + \delta\omega) + (\omega_o + \delta\omega)^2 - k^2 {c_o}^2 = 0$$

As $\tau\omega \ll 1$ and $\tau\delta\omega \ll 1$, we can neglect small terms:

$$\begin{aligned} i\tau(\omega_o)^3 - i\tau k^2 {c_1}^2(\omega_o) + 2\omega_o\delta\omega + {\omega_o}^2 - {c_o}^2 k^2 &= 0 \\ i\tau(\omega_o)^3 - i\tau k^2 {c_1}^2(\omega_o) + 2\omega_o\delta\omega &= 0 \\ \delta\omega &= \frac{1}{2} i\tau(k^2 {c_1}^2 - {\omega_o}^2) \\ &= \frac{1}{2} i\tau(k^2({c_1}^2 - {c_o}^2) \\ &= \frac{1}{2} i\tau\omega(kc_o)\left(\frac{{c_1}^2}{{c_o}^2} - 1\right) \\ \rightarrow \delta\omega &= i\tau\pi\omega\left(\frac{{c_1}^2}{{c_o}^2} - 1\right) \end{aligned}$$

Now considering the form of ϕ:

$$\begin{aligned} \phi &= a e^{i(\omega t - kx)} \\ &= a e^{i((\omega_o + \delta\omega)t - kx)} \\ &= a e^{i\delta\omega t} \cdot e^{i(\omega_o t - kx)} \end{aligned}$$

The amplitude decay term is thus:

$$\begin{aligned} a_d &= e^{i\delta\omega t} \\ &= e^{i\left(i\tau\pi\omega\left(\frac{{c_1}^2}{{c_o}^2} - 1\right)\right)t} \\ &= e^{-\left(\tau\pi\omega\left(\frac{{c_1}^2}{{c_o}^2} - 1\right)\right)t} \end{aligned}$$

Note that this is only the first order approximate solution to the problem.