Additional Problems

Dynamics

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Introduction

1: When a fluid flows through a pipe, because of its viscosity the speed is greatest at the centre of the pipe and approaches zero at the walls i.e. there is a velocity gradient in the pipe perpendicular to the direction of flow. There are shear forces (perpendicular to the direction of flow) between the layers of fluid moving past each other at different speeds — these forces are called viscous forces. As a consequence of these viscous forces, a pressure gradient is required to drive a fluid through a pipe. The **viscosity** of a fluid is defined as the shear force per unit area per unit velocity gradient.

(a) At low speeds the flow of fluid in a pipe is **laminar**: the stream lines are steady and parallel to the pipe. The pressure gradient in the pipe $\mathrm{d}P/\mathrm{d}x$ is then related to the mean velocity $\bar{v}$, the viscosity η and the diameter d of the pipe. Use the method of dimensions to show that $\mathrm{d}P/\mathrm{d}x \propto \eta \bar{v} d^{-2}$.

(b) Beyond a critical speed v_c the flow of fluid in a pipe becomes **turbulent**: eddies form, the stream lines are unsteady, and the fluid is subject to accelerations. The pressure gradient is then related, as in (a), to $\bar{v}$, η and d, and in addition to the fluid density, ρ. Show that under these conditions $\mathrm{d}P/\mathrm{d}x \propto (\eta^2/d^3\rho) f(N_\mathrm{Re})$, where $f(N_\mathrm{Re})$ is any function of the dimensionless quantity $N_\mathrm{Re} = \bar{v}/(\eta d^{-1}\rho^{-1})$, called the *Reynold's number*.

Experimental Physics

2: In a historic piece of research, James Chadwick in 1932 obtained a value for the mass of the neutron by studying elastic collisions of fast neutrons with nuclei of hydrogen and nitrogen. He found that the maximum recoil velocity of hydrogen nuclei (initially stationary) was 3.3×10^7 m s^{-1}, and that the maximum recoil velocity of nitrogen 14 nuclei was 4.7×10^6 m s^{-1} with an uncertainty of $\pm 10\%$. What does this tell you about

(a) the mass of a neutron,

(b) the initial velocity of the neutrons used?

(Take the uncertainty of the nitrogen measurement into account. The error calculation is complicated — it is simplest in this case to repeat the calculation of the neutron mass using 4.2×10^6 m s^{-1} (i.e. the nitrogen recoil velocity minus the error) and then take the difference between this value and the value obtained using 4.7×10^6 m s^{-1}. You can adopt a similar procedure when finding the error in the velocity of the neutrons. Take the mass of an H nucleus as m_u and the mass of a nitrogen 14 nucleus as $14m_\mathrm{u}$.)

3: A rectangular brass bar of mass M has dimensions $a,\ b,\ c$. The moment of inertia I about an axis in the centre of the ab face and perpendicular to it is

$$I = \frac{M}{12}(a^2 + b^2)$$

The following measurements are made:

$$M = 135.0 \pm 0.1 \text{ g},\ a = 80 \pm 1 \text{ mm},\ b = 10 \pm 1 \text{ mm},\ c = 20.00 \pm 0.01 \text{ mm}$$

What is the percentage standard error in
(a) the density ρ of the material and
(b) the moment of inertia I?

Forces

4: From Coulomb's law the force on a charge q_2 due to a charge q_1 is given by

$$\underline{\boldsymbol{F}} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\underline{\hat{\boldsymbol{r}}}$$

where $\underline{\boldsymbol{r}}$ is the vector from q_1 to q_2 and ε_0 is the permittivity of free space; the potential energy of q_2 due to q_1 is given by

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

A charge q is placed at each of the four corners ($\pm a$, 0, 0), (0, $\pm a$, 0) of a square. A charge Q of the same sign as q is placed at the centre of the square.

(a) Use Coulomb's law and vector addition to show that Q is in equilibrium at the point (0, 0, 0).

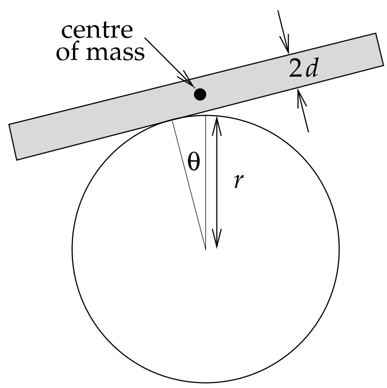
(b) Find an expression for the potential energy of Q at a point (x, y, z) near the origin; hence show that for very small displacements from the origin, i.e. if $(x^2 + y^2 + z^2)^{1/2} \ll a$, the potential energy is given by

$$U(x,y,z) = \frac{Qq}{4\pi\varepsilon_0 a}\left[4 + (x^2 + y^2 - 2z^2)/a^2 + \ldots\right].$$

(c) Using the above expression for $U(x,y,z)$, show that the equilibrium at the centre of the square is stable against a small displacement in the plane of the square, but is unstable for a displacement normal to this plane.

(d) What happens if Q has the opposite sign to q?

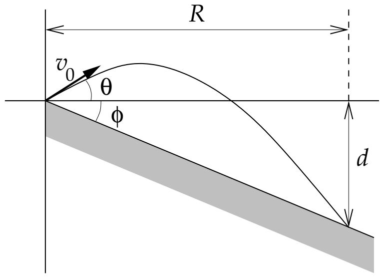
5: A uniform plank of thickness $2d$ and weight W is balanced horizontally across the top of a fixed cylinder of radius r, whose axis is horizontal and perpendicular to the length of the plank. Prove that the gain of potential energy when the plank is turned without slipping through an angle θ in a vertical plane is $W[r\theta \sin\theta - (r+d)(1-\cos\theta)]$ and deduce the condition for stability.



6: A rope is wound round a fixed cylinder of radius r so as to make n complete turns. Show that if one end of the rope is held by a force F, a force $F\exp(2\pi n\mu)$ must be applied to the other end to produce slipping, where μ is the coefficient of friction between the rope and cylinder. The cylinder is now released, the force on one end of the rope is F and the force on the other end is such that the rope is on the point of slipping; find the work required to turn the cylinder through one complete turn under these conditions.

Kinematics

7: A projectile is shot from the origin with initial velocity v_0 and inclination angle θ as shown.



Show the following:

(a) the range R, v_0, θ and the drop d are related by

$$R\sin 2\theta + d(1+\cos 2\theta) = R^2/R_0$$

where

$$R_0 \equiv v_0^2/g;$$

(b) the condition for maximum range R_m is

$$\tan 2\theta_m = R_m/d;$$

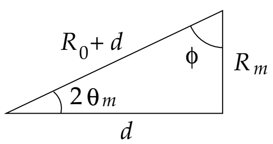
(c) if the land falls off with a constant slope angle ϕ then the maximum range angle θ_m and ϕ are related by

$$2\theta_m + \phi = 90^\circ;$$

(d) the maximum range is given by

$$R_0^2 + 2dR_0 = R_m^2;$$

(e) the optimum angle, maximum range, slope angle ϕ and the elevation drop d satisfy the triangle relation shown.



Check that the answers to (b), (c) and (d) agree with the results obtained when $\phi = 0$, as in the lecture notes and in question 12 on the Part IA Physics problem sheet.

8: Due to air resistance a rifle bullet has a deceleration bv^3, where v is its speed and b is a constant. Show that v, the time t, and the distance z travelled by the bullet are related by the equations

$$t = \frac{1}{2}bz^2 + z/u \quad \text{and} \quad v = u/(1+bzu)$$

where u is the initial speed of the bullet.

Dynamics

9: Two buckets of water, each of total mass M, are attached to the ends of a light cord which passes over a smooth pulley of negligible mass. The buckets are at rest. Water starts to leak from one bucket at a constant rate k. Find the equations of motion for each bucket. If v and m are the velocity and mass of the leaky bucket at a given time, deduce that

$$\frac{\mathrm{d}v}{\mathrm{d}m} = \frac{g(m-M)}{k(m+M)}.$$

If εM $(0 < \varepsilon < 1)$ is the mass of water in this bucket initially, determine its speed at the instant when the bucket becomes empty.

10: The following expression gives the resistive force exerted on a sphere of radius r moving at speed v through air. It is valid over a very wide range of speeds.

$$R(v) = 3.1 \times 10^{-4} rv + 0.87 r^2 v^2$$

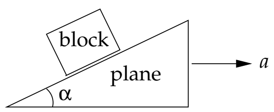
where R is in N, r in m and v in m s^{-1}. Consider water drops falling under their own weight and reaching a terminal speed.

(a) For what range of values of r is the terminal speed determined within 1% by the first term alone in the expression for $R(v)$?

(b) For what range of values of larger r is the terminal speed determined within 1% by the second term alone?

(c) Calculate the terminal speed of a raindrop of radius 2 mm. If there were no air resistance, from what height would it fall from rest before reaching this speed?

11: A (rough) inclined plane has a horizontal acceleration a to the right as shown. Show that the block will slide on the plane if $a > g\tan(\theta - \alpha)$ where $\mu_s = \tan\theta$ is the coefficient of static friction for the contacting surfaces.



12: A light ball of mass m is released from rest from a height h and falls under gravity. The air resistance is roughly proportional to, and in the opposite direction to, the speed v with which it is moving and is given by $-bv$, where b is a constant. Find the equation for the acceleration of the ball.

(a) Describe the subsequent motion of the ball in words and obtain an expression for the speed at which the acceleration is zero — this is called the *terminal speed* or *terminal velocity*.

(b) Find expressions for the speed and height of the ball at time t in terms of b, m, g the acceleration due to gravity, and h.

(c) If the ball reaches a terminal speed of 10.0 m s^{-1}, (i) how long does it take for the speed to reach 9.5 m s^{-1} and (ii) how far will it have fallen when it reaches this speed?

(d) Find the instantaneous power expended against the air resistance at time t; hence obtain an expression for the work done against the air resistance over the period $t = 0$ to $t = \tau$. By considering the loss in gravitational potential energy and the kinetic energy gained over the same time interval, show that your expression is consistent with conservation of energy.

13: In question 17 on the Part IA Physics problem sheet a horizontal force F is applied to a block of mass m resting on top of a slab of mass M which itself rests on a frictionless floor. Use the notation and results from that question to consider energy changes over a period from $t = 0$ to $t = \tau$, assuming that at time $t = 0$ both the block and slab are at rest.

(a) Find the work done by F over the period $t = 0$ to $t = \tau$ and compare this with the total kinetic energy of the block and slab for

(i) $F < F_{\text{lim}}$;

(ii) $F > F_{\text{lim}}$ (assume that the block does not reach the end of the slab in the time $t = 0$ to $t = \tau$).

(b) For $F > F_{\text{lim}}$ find the work done against friction explicitly and using the results from (a)(ii) show that your expression is consistent with conservation of energy.

14: A rocket of total mass M contains fuel of mass εM $(0 < \varepsilon < 1)$. When ignited the fuel burns at a constant mass-rate k, ejecting exhaust gases with constant speed v_0 with respect to the rocket. Assuming no other forces act on the rocket, find the distance travelled when the rocket has just used up all its fuel, assuming it starts from 'rest'.

Frames of reference (and collisions)

15: A particle of mass m_1 and initial velocity u_1 strikes a stationary particle of mass m_2. The collision is perfectly elastic. It is observed that after the collision the particles have equal and opposite velocities. Find

(a) the velocity of the zero momentum frame;

(b) the ratio m_1/m_2;

(c) the total kinetic energy of the two particles in the zero momentum frame expressed as a fraction of $\frac{1}{2}m_1u_1^2$;

(d) the final kinetic energy of m_1 in the lab frame as a fraction of $\frac{1}{2}m_1u_1^2$.

16: Three perfectly elastic bodies of masses $5m$, m, $5m$ are arranged in that order on a straight line, and are free to move along it. Initially the middle one is moving with velocity v, and the others are at rest. Find how many collisions take place in the subsequent motion, and verify that the final value of the kinetic energy is equal to the initial value.

17: A particle of mass m_1 moving with velocity $\underline{\boldsymbol{u}}$ in the laboratory frame collides, elastically, with a particle of mass m_2 initially at rest; after the collision in the laboratory frame the particles move making angles θ_1 and θ_2 with the original direction of m_1. Use the zero-momentum frame to investigate the following.

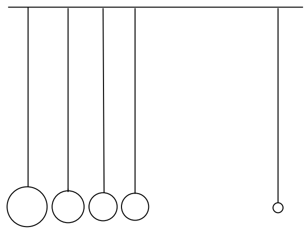
(a) If $m_1 > m_2$, show that the maximum value of θ_1 is given by $\sin\theta_{\text{max}} = m_2/m_1$.

(b) If $m_1 \ll m_2$, show that $\theta_1 \approx \pi - 2\theta_2$.

(c) Show explicitly that if energy is conserved in the zero-momentum frame it is conserved in the laboratory frame.

(d) Find an expression for the fraction of the initial kinetic energy of m_1 lost to m_2 in the collision in the laboratory frame. Find the value of m_2 for which the fractional loss is a maximum. Sketch the fractional loss as a function of (i) m_2/m_1 for fixed θ_2 and (ii) θ_2 for fixed m_2/m_1.

18: A collision apparatus is made of a set of n graded masses suspended so that they are in a horizontal line and not quite in contact with each other. The first mass is fm_0, the second is f^2m_0, the third f^3m_0, and so on, so that the last mass is f^nm_0. The first mass is struck by a particle of mass m_0 travelling at a speed v_0. This produces a succession of collisions along the line of masses.



(a) Assuming that all the collisions are perfectly elastic, show that the last mass flies off with a speed v_n given by

$$v_n = \left(\frac{2}{1+f}\right)^n v_0.$$

(b) Hence show that, if f is close to unity, so that it can be written as $1 \pm \varepsilon$ (with $\varepsilon \ll 1$), this system can be used to transfer virtually all the kinetic energy of the incident mass to the last one, even for large n.

(c) For $f = 0.9$, $n = 20$, calculate the mass, speed and kinetic energy of the last mass in the line in terms of the mass, speed and kinetic energy of the incident particle. Compare this with the result of a direct collision between the incident mass and the last mass in the line.

Numerical and some algebraic answers

2: (a) $m_\mathrm{n} = (1.2 \pm 0.3)\ m_\mathrm{u}$; (b) $v_\mathrm{initial} = (3.1 \pm 0.4) \times 10^7$ m s^{-1}.
3: (a) 10%; (b) 2.5%.
5: $d < r$.
6: $2\pi r F(\exp(2\pi n\mu) - 1)$.
9: $Mg\left(-2\ln\left[1-\frac{1}{2}\varepsilon\right] - \varepsilon\right)/k$.
10: (a) $r \leq 0.03$ mm; (b) $r \geq 2$ mm; (c) 9.7 m s^{-1}; 4.8 m.
12: $\ddot{x} = g - (bv/m)$; (a) $v = mg/b$; (b) $v = v_\mathrm{t}(1-\mathrm{e}^{-\alpha t})$, height $= h - v_\mathrm{t}\left(t - \frac{1}{\alpha}(1-\mathrm{e}^{-\alpha t})\right)$,
where $v_\mathrm{t} = mg/b$, $\alpha = b/m$; (c) (i) 3.05 s, (ii) 20.8 m; (d) $W = bv_\mathrm{t}^2\left[\tau + \frac{1}{\alpha}\left(2\mathrm{e}^{-\alpha\tau} - \frac{1}{2}\mathrm{e}^{-2\alpha\tau} - \frac{3}{2}\right)\right]$.
14: $z = (v_0 M/k)[\varepsilon + (1-\varepsilon)\ln(1-\varepsilon)]$.
15: (b) 1/3; (c) 3/4; (d) 1/4.
16: 3; final velocities: $-2v/9$, $7v/27$, $10v/27$.
18: $m_n = 0.12m_0$, $v_n = 2.8v_0$, $\mathrm{KE}_n = 0.95\ \mathrm{KE}_\mathrm{init}$, $\mathrm{KE}_\mathrm{direct} = 0.38\ \mathrm{KE}_\mathrm{init}$.

If you find any errors please contact the lecturer (Jeremy Baumberg, jjb12@cam.ac.uk).