OSCILLATING SYSTEMS

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Additional Examples

Simple harmonic motion

1. The mutual potential energy of two ions (H^+ and Cl^-) a distance r apart is given by

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{B}{r^9}.$$

At equilibrium, $r = 0.13\,\mathrm{nm}$; find the effective spring constant for small displacements from equilibrium. Hence estimate the frequency of vibration of the molecule.

(**Ans:** $840\,\mathrm{N\,m^{-1}}$, $\approx 1.1 \times 10^{14}\,\mathrm{Hz}$)

2. A mass M is suspended at the end of a spring of spring constant k. The mass of the spring is m and the displacement of each element of the spring is proportional to its distance from the fixed end of the spring. Show that the total kinetic energy of the spring is $\frac{1}{6}mv^2$ where v is the velocity of the suspended mass M. Hence, by considering the total energy of the oscillating system, show that the frequency of oscillation is given by

$$\omega^2 = \frac{k}{M + m/3}.$$

3. The equation of motion of a simple pendulum is given by

$$\ddot{\theta} = -\omega_0^2 \sin\theta \approx -\omega_0^2 \theta$$

for very small angular displacements θ. For slightly larger displacements we retain the second term in the Taylor expansion of $\sin\theta$ giving a non-linear equation of the form

$$\ddot{\theta} \approx -\omega_0^2\theta + \omega_0^2 \frac{1}{3!}\theta^3.$$

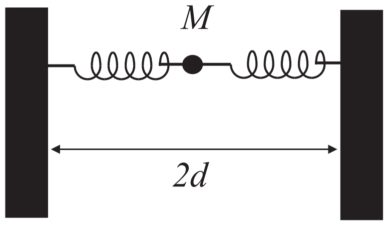
Show that this equation has a solution of the form

$$\theta(t) \approx A(\cos\omega t + \epsilon\cos 3\omega t)$$

where $\epsilon \ll 1$ and $\frac{1}{6}A^2 \ll 1$, and find approximate values for ω and ϵ.

(**Ans:** $\omega \approx \omega_0\left(1 - \frac{1}{16}A^2\right)$, $\epsilon \approx -\frac{1}{192}A^2$)

4. The diagram below shows a mass M supported by 2 springs between 2 vertical pillars. Assuming the springs each have natural length d and spring constant k, determine whether vertical SHM is possible for the system and if so determine the natural frequency ω.



(**Ans:** $\omega^2 = \frac{2k}{m}(1 - \cos^3\phi)$, where ϕ is the angle that the spring makes to the horizontal when in equilibrium.)

Damped simple harmonic motion

5. The equation of motion for a damped simple harmonic oscillator is

$$m\ddot{x} + b\dot{x} + kx = 0$$

where b and k are constants, m is the mass and x is the displacement of the system.

Describe the conditions for lightly damped, critically damped and over-damped oscillations. Draw diagrams to show how displacement varies with time in the three cases, given that in each case the system is initially displaced and then released from rest.

A system whose natural frequency in the absence of damping is $4\,\mathrm{rad\,s^{-1}}$ is subject to a damping force such that $b/m = 10\,\mathrm{s^{-1}}$. Show that the system is over-damped and that the general solution for the displacement is

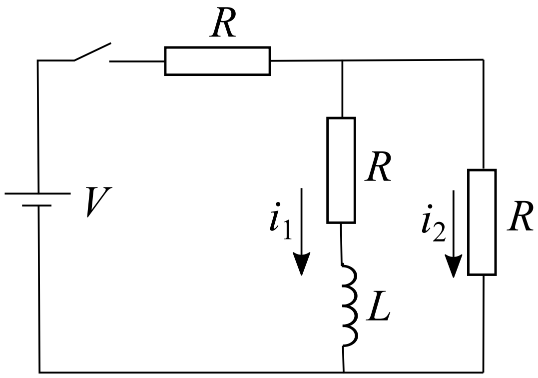
$$x = A\mathrm{e}^{-2t} + B\mathrm{e}^{-8t}.$$

The mass is initially at $x = +0.5\,\mathrm{m}$ and given an initial velocity V *towards* the equilibrium position. Find the smallest value of V that will produce a negative displacement.

(**Ans:** $V > 4\,\mathrm{m\,s^{-1}}$)

Electrical circuits

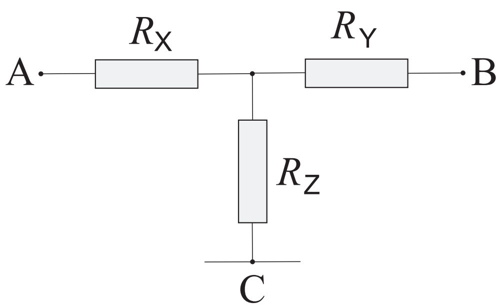
6. In the circuit shown below the inductor has a self inductance L and the three resistors have the same resistance R. The switch is closed at $t = 0$. Obtain expressions for the currents i_1 and i_2 as a function of time, and illustrate these variations on a labelled graph.



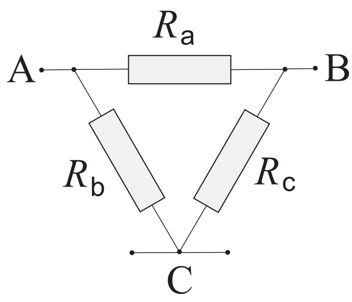
(**Ans:** $i_1 = (V/3R)(1 - \mathrm{e}^{-3Rt/2L})$, $i_2 = (V/3R)(1 + \frac{1}{2}\mathrm{e}^{-3Rt/2L})$)

7. Circuit (a) below is known as a "Star" network and circuit (b) a "Delta" network. Using simple circuit theory, find expressions for values of $R_\mathrm{X}, R_\mathrm{Y}$ and R_Z in terms of $R_\mathrm{a}, R_\mathrm{b}$ and R_c such that both circuits will behave as if they are electrically identical, when viewed via terminals A, B and C.

a)

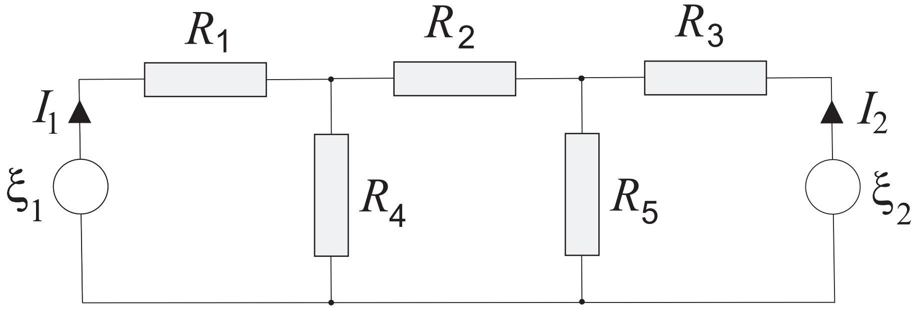


b)



(**Ans**: $R_X = \frac{R_A R_B}{R_A+R_B+R_C}$, $R_Y = \frac{R_A R_C}{R_A+R_B+R_C}$, $R_Z = \frac{R_B R_C}{R_A+R_B+R_C}$.)

8. (a) Write down the equations needed to solve the circuit below using the Loop Current (Kirchhoff's Voltage Law) method. (You are not required to solve them!).



(b) How many equations would be required if Nodal Analysis (Kirchhoff's Current Law) was used?

(c) Using the transformation obtained in the previous question, show that the circuit can be reduced to having only one unknown potential.

If $R_1 = 1\Omega$, $R_2 = R_4 = R_5 = 3\Omega$, $R_3 = 2\Omega$, $\xi_1 = 22V$, and $\xi_2 = 11V$ determine the currents I_1 and I_2 flowing in each of the two batteries in the circuit.

Oscillations in electrical circuits and complex impedance

9. A resonant circuit comprises an inductor, a capacitor and a resistor, all three components being in parallel. At time $t = 0$ the current in the inductor has a maximum value and equals one ampere. Show that a voltage across the circuit of the form

$$V = \frac{1}{\omega C} \mathrm{e}^{-bt} \sin \omega t$$

satisfies the appropriate differential equation and find b and ω in terms of the component values, L, C and R.

In the above circuit the components have the values 0.1 H, 10 μF and $10^5\,\Omega$. Calculate

(i) the initial energy stored in the circuit

(ii) the average power dissipated in the resistor during the first few cycles and

(iii) the time taken for the r.m.s. voltage to fall to by a factor of 2.

(**Ans:** (i) 0.05 J (ii) 0.05 W (iii) 1.4 s)

10. An electrical circuit consists of a resistance R, inductance L and capacitance C in series. If a charge is put on the capacitor at some instant, determine the condition that V_{C}, the voltage across the capacitor, is subsequently oscillatory. Assuming the condition is satisfied, derive an expression for the time T for the amplitude of V_{C} to drop by a factor of e.

An external voltage source of variable frequency is introduced into the circuit in series with the other components. Show that, for small R, the width of the resonance (defined as the angular frequency range for which the amplitude of V_{C} is greater than $1/\sqrt{2}$ of its resonance value) is approximately $2/T$.

(**Ans:** $R^2 < 4L/C$, $T = 2L/R$)