OSCILLATING SYSTEMS

N. C. Greenham

Examples Sheet

There are 24 problems on this Examples Sheet covering the 12 lectures on Oscillating Systems in the Lent Term.

At the end of this sheet a number of tripos questions relating to the material covered in this part of the course are also listed for you to try. You may find these helpful to give you practice in answering Tripos standard questions and for revision purposes. [Questions numbered with an 'A' prefix would each need to be answered in about 10 minutes under examination conditions and those labeled with 'B' or 'C' prefixes would need to be answered in around 25-30 minutes each].

For those who are interested, there are some **strictly optional** additional problems – less structured and usually of a slightly more mathematical nature – which can be downloaded from TiS.

If you find any errors please contact the lecturer, Prof. Neil Greenham (ncg11@cam.ac.uk).

Simple harmonic motion

1. A 1.0 kg object is attached to a horizontal spring. The spring is initially stretched by 0.10 m, and the object is released from rest there. It proceeds to move without friction. The next time the speed of the object is zero is 0.5 s later. What is the maximum speed of the object?

2. (a) A mass m is supported from a light spring with spring constant k. The mass is displaced slightly from its equilibrium position. Show that it will undergo vertical oscillations of period $T = 2\pi\sqrt{m/k}$. Explain why the gravitational acceleration g does not enter into the expression for T.

   (b) Describe the changes in the kinetic energy and the elastic and gravitational potential energies of the system as the mass oscillates up and down.

   (c) What will the period become if the length of the spring is halved ?

   (d) What will the period become if the two half-springs are used in parallel ?

   (e) An astronaut on the surface of the moon weighs rock samples using a light spring balance. The balance, which was calibrated on earth, has a scale 100 mm long which reads from 0 to 1.0 kg. The astronaut observes that a certain rock gives a steady reading of 0.40 kg and, when disturbed, vibrates with a period of 1.0 s. What is the acceleration due to gravity on the moon?

3. A simple pendulum consisting of a mass m on a string of length l is released from rest from an angle of θ_0 to the vertical.

   (a) Assuming that the pendulum undergoes simple harmonic motion, find an expression for its angular displacement as a function of time; hence determine its speed as it passes through the equilibrium position.

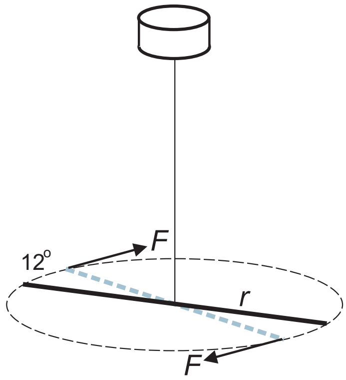
   (b) Using conservation of energy find an exact expression for this speed.

   (c) Show that your results for (a) and (b) are the same when θ_0 is small.

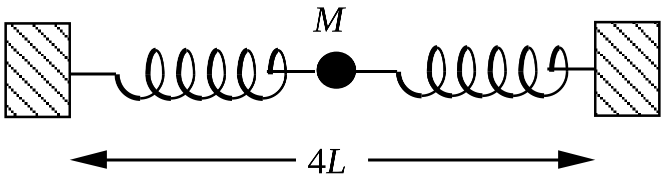
   (d) Find the percentage difference between the two results in (a) and (b) when $\theta_0 = 0.20$ rad.

   (e) For the SHM case, find the tension T in the pendulum string during the motion. *Hint: the mass at the end of the pendulum is instantaneously doing circular motion with angular velocity* $\dot{\theta}$.

4. A torsional pendulum disc is suspended in the horizontal plane by a vertical wire attached at its centre as shown in the diagram. When the disk is twisted to an angle of 12° the wire stores 0.52 J of energy. When the disc is released, it oscillates as a torsional pendulum with a period of 0.5 s. Determine its moment of inertia.



5. A mass M is supported by a smooth table and connected by two light horizontal springs to two fixed blocks as shown in the diagram below. Each spring is of natural length L and has a spring constant k.



(a) Derive an expression for the angular frequency ω of horizontal longitudinal oscillations of the mass M.

(b) When the mass M is given a *small* transverse displacement, y (such that $y \ll L$), show that the transverse restoring force is approximately ky. Hence derive an expression for the angular frequency ω of small transverse oscillations of the mass M.

(c) Describe qualitatively the motion that results if the mass M is displaced slightly in an arbitrary direction and released from rest.

6. Two masses, m_1 and m_2, are connected by a spring of force constant k and are acted on by no other forces. They are oscillating about their centre of mass. Write down expressions for the potential and kinetic energies of the system and hence show that the frequency of oscillation of the system is

$$f = \frac{1}{2\pi}\left[k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right]^{1/2}.$$

The ^{12}C and ^{16}O atoms in carbon monoxide are held together by a covalent bond; their equilibrium separation is 0.130 nm. For a small change, u, in the separation of the atoms from their equilibrium position the bond behaves like a spring in that the potential energy change is proportional to u^2. By plotting the appropriate graph, use the data below to test the validity of this assertion and estimate the effective value of k for the CO molecule. Hence find the frequency of vibration of the CO molecule.

u / nm	0.001	0.002	0.003	0.004	0.005
PE / eV	0.006	0.023	0.057	0.093	0.150

[The PE may be assumed to be zero at the equilibrium separation. Assume a random error of 0.002 eV in the PE measurements and that the error in u is negligible.]

7. The response of a system is given by

$$x = A\cos\omega_1 t + A\cos\omega_2 t$$

where $\omega_1 = 6\pi\ \mathrm{s}^{-1}$ and $\omega_2 = 4\pi\ \mathrm{s}^{-1}$.

(a) Sketch and label the phasor diagrams representing this motion at time $t = 0$, $t = 0.1$ s, $t = 0.2$ s, $t = 0.3$ s and $t = 0.4$ s.

(b) Rewrite the expression in terms of the product of two cos terms. Hence find a general expression for the times at which the response is zero; relate this to your phasor diagrams. Sketch the response as a function of time.

Oscillations in a a potential energy well

8. (a) A particle of mass m moves in a potential $\frac{1}{2}kx^2$. Use conservation of energy to show that the particle undergoes SHM. What is the frequency of the SHM?

(b) The previous potential is removed, and the particle now bounces back and forth between (impenetrable) walls at $x = 0$ and $x = l$. The particle's collisions with the walls are elastic. If the particle has kinetic energy E, show the frequency of the oscillations varies as $\nu \propto \sqrt{E}$. Sketch the potential well the particle is oscillating in.

(c) The particle is now subject to a gravitational field g and dropped from rest a distance h above the floor. When it hits the floor it bounces elastically, so energy is conserved. What is the maximum kinetic energy of the particle, E? Show the frequency of the oscillations varies as $\nu \propto 1/\sqrt{E}$.

(d) The same particle moves in a general potential $V(x)$ with a minima at $x = 0$. Show that if the particle is released close to this minimum, it will undergo SHM around the minimum with frequency $\omega_0^2 = V''(0)/m$. In this case, what is the relationship between the maximum kinetic energy, E, and the frequency of the oscillation?

(e) What distinguishes the oscillations in (b) and (c) from SHM? Why does the general principle in (d) not apply to the potentials in (b) and (c)?

Hint: A barrier such as a wall or a floor can be thought of as a a region of infinite potential energy.

Complex representation of SHM

9. It is possible to describe a system performing SHM in four ways:

$$x = A\cos(\omega t + \phi)$$
$$x = B_1 \cos\omega t + B_2 \sin\omega t$$
$$x = C\mathrm{e}^{\mathrm{i}\omega t} + C^*\mathrm{e}^{-\mathrm{i}\omega t}$$
$$x = \mathrm{Re}\{D\mathrm{e}^{\mathrm{i}\omega t}\}$$

where A, ϕ, B_1 and B_2 are real and $C(= C_1 + \mathrm{i}C_2)$ and $D(= D_1 + \mathrm{i}D_2)$ are complex. Show that these four forms are equivalent and express A and ϕ in terms of (i) B_1 and B_2, (ii) C_1 and C_2 and (iii) D_1 and D_2.

If $\omega = 3.0\,\mathrm{rad\,s^{-1}}$ and $x = 0.020\,\mathrm{m}$ and $\dot{x} = 0.060\,\mathrm{m\,s^{-1}}$ at $t = 0$, find the values of A, ϕ, B_1, B_2, C and D.

Damped Harmonic Motion

10. (a) A pendulum consists of a spherical bob of radius r and density ρ attached to one end of a thin, light rod of length l ($\gg r$) freely pivoted at its other end. Derive the equation of motion for the angular displacement θ of the rod from the vertical, and show that the angular frequency of small oscillations of the pendulum is given by $\omega_0 = \sqrt{g/l}$.

(b) The entire pendulum is now immersed in a light fluid which exerts a drag force on the bob of magnitude $\alpha r v$, where v is its speed and α is a constant. The system is displaced from equilibrium and then released. Ignoring buoyancy effects, show that the equation of motion for θ is now given by

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta = 0.$$

where $\gamma = 3\alpha/8\pi r^2\rho$.

Substitute a solution of the form $\theta = A\mathrm{e}^{pt}$ into this equation. Show that the system will oscillate after being released (i.e. it is lightly damped) if $r > (3\alpha/8\pi\rho\omega_0)^{1/2}$, and find the general solution for θ under these circumstances.

If the density of the bob is $4000\,\mathrm{kg\,m^{-3}}$, $l = 1.0\,\mathrm{m}$, $r = 0.020\,\mathrm{m}$ and $\alpha = 2.0\,\mathrm{N\,s\,m^{-2}}$, verify that the pendulum will oscillate after it is displaced from equilibrium and then released. Find (i) the angular frequency of oscillation of the damped pendulum and (ii) how long it takes for the energy to drop by a factor of e. Hence obtain a value for the quality factor, Q, of the oscillation.

11. (a) The pendulum from the previous question is displaced to a small angle θ_0 and released from rest. What ranges of γ give rise to heavy, critical and light damping? Find and sketch $\theta(t)$ in each case.

(b) With each type of damping, $\theta(t)$ is built from terms that decay exponentially. For each damping regime find an expression for τ, the characteristic decay time of the slowest decaying term in $\theta(t)$. Sketch a graph of the τ against γ, and indicate the light, critical and heavy damping regions. What is special about critical damping?

(c) The heavy damping solution has a second faster decaying term. Find its characteristic decay time and add a line to the graph for this decay time as a function of γ.

Forced oscillations

12. A 2.0 kg object attached to a spring moves without friction ($b = 0$) and is driven by an external force given by the expression $F = 3.0 \sin(2\pi t)$, where F is in newtons and t is in seconds. The force constant of the spring is $20.0\,\mathrm{N\,m^{-1}}$. (a) Find the resonance angular frequency of the system. Assuming the system is oscillating in its steady state find (b) the angular frequency of the driven system, and (c) the amplitude of the motion.

Aside: since the system is undamped, the steady state assumption is actually dubious since the transient solutions never vanish. You will learn more about this at IB.

13. **A more challenging problem*.*

An oscillating force $F \cos \omega t$ $(= Re\{F\mathrm{e}^{\mathrm{i}\omega t}\})$, where F is real, is applied to a mass m on the end of a spring of force constant k. The displacement, x, of the particle can be written as $Re\{z\}$. Explain why z obeys the equation

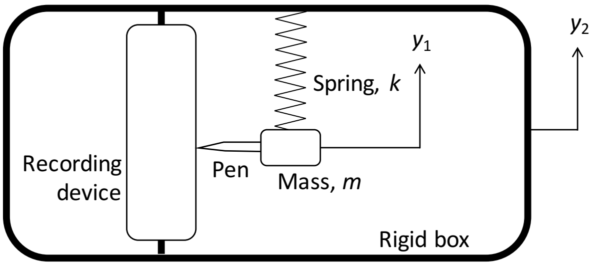
$$m\ddot{z} = -kz + F\mathrm{e}^{\mathrm{i}\omega t}.$$

Show that $z = A\mathrm{e}^{\mathrm{i}\omega t}$ is a solution of the equation and find an expression for A; deduce the expression for the displacement, x. Find the expressions for $\dot{z}$ and $\ddot{z}$ and from these deduce the velocity and acceleration of the mass.

Illustrate on Argand diagrams the complex amplitudes of the applied force and the displacement, velocity and acceleration of the mass when (i) $\omega^2 < k/m$ and (ii) $\omega^2 > k/m$.

What happens when $\omega^2 = k/m$?

14. **A modified recent tripos question **
A seismograph allows the measurement of the amplitude of seismic waves of different frequencies. It can be modelled as a mass m suspended inside a rigid box by a massless spring of constant k, The motion of the mass is damped by a vertical viscous force which may be assumed to be of magnitude $b\,\dot{y}_1$, where y_1 is the height of the mass in the inertial frame of reference of the surface of the Earth (before any seismic wave is present) and b is a damping coefficient. A light frictionless pen records the motion of the mass relative to the box as shown in the figure below.



A seismic wave moves the box vertically so that its height as a function of time is given by $y_2 = a_s \cos(\omega t)$.
(a) Show the differential equation describing the motion of the mass is

$$\ddot{y}_1 + \frac{b}{m}\dot{y}_1 + \frac{k}{m}y_1 = \frac{k}{m}a_S \cos(\omega t).$$

(b) Show the steady state motion is the mass is of the form $y_1 = \mathrm{Re}\left(A(\omega)\mathrm{e}^{\mathrm{i}\Phi(\omega)}\mathrm{e}^{\mathrm{i}\omega t}\right)$, and find expressions for $A(\omega)$ and $\Phi(\omega)$.

(c) Sketch the amplitude of the response, $A(\omega)$, to seismic waves of fixed amplitude a_s as a function of the angular frequency ω for the case where b is small. Use this sketch to explain why seismographs are designed to work in the high-frequency domain.

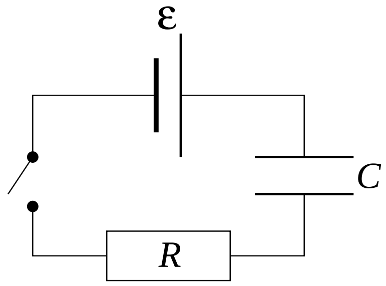
Electrical circuits

15. A battery with emf $\mathcal{E}$ and internal resistance r is connected to a pair of resistors of resistance R_1 and R_2.

(a) The two resistors are connected in series. Find expressions for (i) the current supplied by the battery, (ii) the current through R_1, (iii) the power produced by the battery and (iv) the power dissipated in R_1.

(b) Repeat the calculations in (a) when the two resistors are connected in parallel. Assuming that the internal resistance of the battery $r = 0$, sketch a graph showing how the currents through R_1 and R_2 and the current supplied by the battery vary as a function of R_1.

16. Consider the circuit shown below. The capacitor is initially uncharged; at time $t = 0$ the switch is closed.



(a) Describe, without using detailed mathematics, what happens to the charge on the capacitor and the current flowing in the circuit with time after the switch is closed; discuss the energy changes that occur in the capacitor and the battery. When the capacitor is fully charged, how much energy is stored on the capacitor and how much work has been done by the battery? Hence deduce the total energy dissipated in the resistor during the process of charging.

(b) Explain why the equation for the charge, q, on the capacitor after $t = 0$ is given by

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = \mathcal{E}.$$

(c) Show, by direct substitution into the above equation or otherwise, that the solution is given by

$$q = A + B\mathrm{e}^{-t/\tau}$$

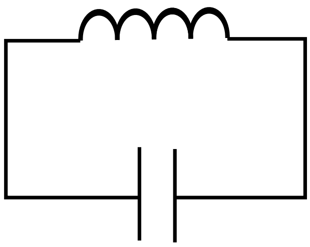
and hence find expressions for A and τ. By considering the appropriate boundary condition find an expression for B. Find the corresponding expression for the current flowing in the circuit. Sketch graphs showing the variation of the charge on the capacitor and the current in the circuit as functions of time.

(d) Use your expression for the current to verify that the total charge which has flowed in the circuit when the capacitor is fully charged is $C\mathcal{E}$.

(e) Use your expression for the current to find the instantaneous power dissipated in the resistor and hence find the total energy dissipated in the resistor as the capacitor is charged. Check that this agrees with your deduction in (a).

Oscillations in electrical circuits

17. A 5-μF capacitor is connected across a 10-mH inductor as shown. At time $t = 0$ the capacitor is uncharged and there is a current of 10 mA in the inductor.



(a) Describe in words the subsequent behaviour of the system.

(b) Write down the differential equation obeyed by the charge on the capacitor; deduce the frequency of oscillation of the circuit and, taking account of the initial conditions, find expressions for the charge on the capacitor and the current flowing in the circuit.

(c) What is the total energy stored in the system? Find expressions for the energy stored in the inductor and in the capacitor and sketch a graph showing how the energy stored in the inductor and capacitor varies as a function of time.

Complex impedance and resonance in electrical circuits

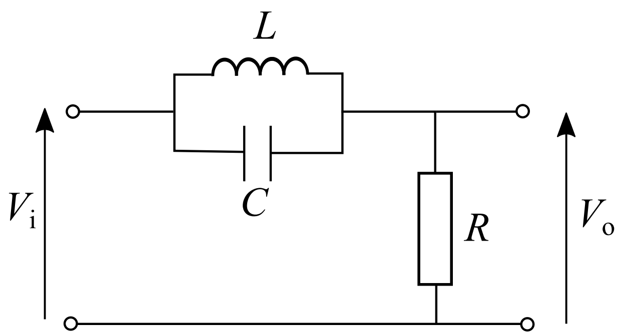
18. (a) Explain what is meant by the terms *resistance*, *inductance*, *capacitance* and *impedance* in relation to AC circuits.

(b) An alternating voltage source, $V_{\text{in}} = V\mathrm{e}^{\mathrm{i}\omega t}$, has amplitude $V = 10\,\mathrm{V}$ and angular frequency $\omega = 300\,\mathrm{rad\,s^{-1}}$. Sketch and label an Argand diagram showing the voltage V_{in} at time $t = 0$ and $t = 0.007\,\mathrm{s}$.

(c) The voltage source, V_{in}, is applied *in turn* across a $10\,\Omega$ resistor (R), a $10\,\mathrm{mH}$ inductor (L) and a $100\,\mu\mathrm{F}$ capacitor (C). Sketch and label, on three separate Argand diagrams, the impedances of R, L and C, and the currents flowing through R, L and C at time $t = 0$ and $t = 0.007\,\mathrm{s}$.

(d) The electronic components are now connected together in the following configurations: (i) R and L in series, (ii) L and C in parallel; and the voltage V_{in} is applied. On two separate Argand diagrams, sketch and label the resulting impedances, and the current flowing through each of the circuits at time $t = 0$ and $t = 0.007\,\mathrm{s}$.

19. An inductor, L, and capacitor, C, in parallel are connected in series with a resistance R as shown in the figure below to form a *band rejection circuit*, which can be used to selectively remove a range of input frequencies from the output. A voltage source, $V_i = Ve^{i\omega t}$, is applied to the circuit.

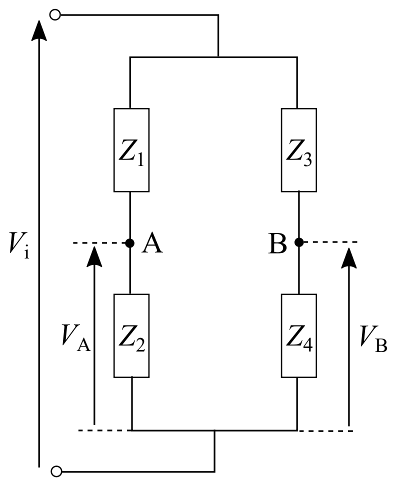


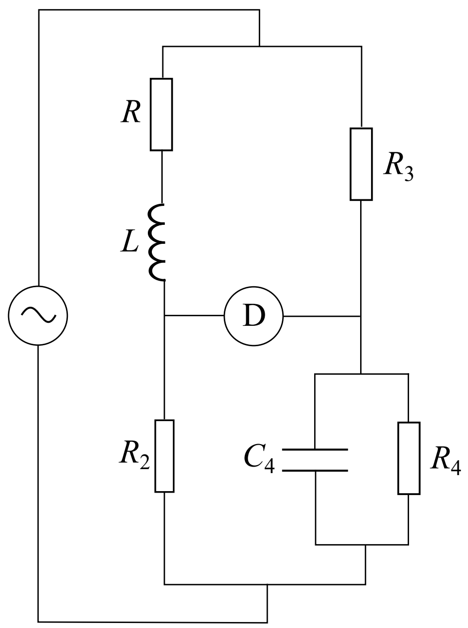
(a) Obtain a general expression for the output voltage, V_o.

(b) For $V = 10\,\text{V}$, $\omega = 300\,\text{rad s}^{-1}$, $R = 10\,\Omega$, $L = 10\,\text{mH}$, $C = 100\,\mu\text{F}$, show the relationship between V_o and V_{in} on an Argand diagram.

(c) The input angular frequency, ω, is varied, while keeping $V = 10\,\text{V}$. Find the value of ω for which the output voltage is zero and sketch the amplitude of V_o as a function of the input frequency.

20. (a) By considering each half of the circuit on the left below as a potential divider, write down expressions for the voltages V_A at A and V_B at B. Hence show that the condition that the points A and B are at the same potential is given by $Z_1/Z_2 = Z_3/Z_4$.

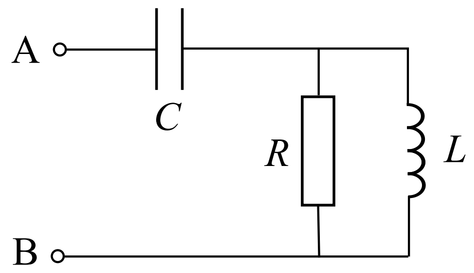




(b) The 'bridge' circuit on the right above is said to be 'balanced' when the detector D registers no voltage difference between its terminals. Use the result derived in (a) to find formulae for R and L in terms of the other components when the circuit is balanced.

(c) A bridge has $R_2 = R_3 = 300\,\Omega$, and unknown R and L. Balance is obtained by adjusting R_4 to $9\,\text{k}\Omega$ and C_4 to $1.0\,\mu\text{F}$. Find the values of R and L.

21. (a) Find the complex impedance of the circuit shown below.



(b) If $L = CR^2$ and a sinusoidal voltage of angular frequency $1/\sqrt{LC}$ is applied across AB, show that the current flowing through the circuit is $\pi/4$ out of phase with the applied voltage. Which leads ?

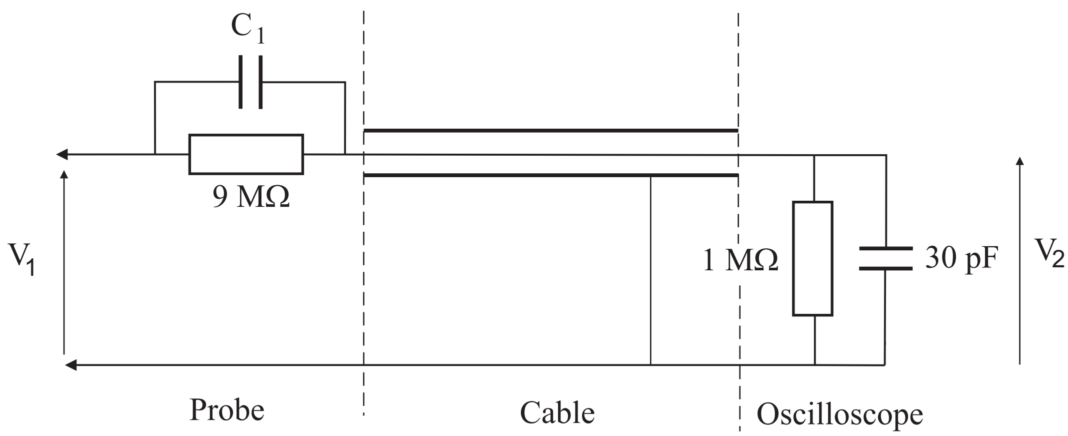
22. a) An AC load consists of two branches connected in parallel. The first is a resistor R connected in series with a an inductor L and the second is a resistor R connected in series with a capacitor C.

Obtain an expression for the total impedance of the load at an angular frequency ω and hence derive the relationship between L, C and R which is necessary for the impedance to be purely resistive at all frequencies.

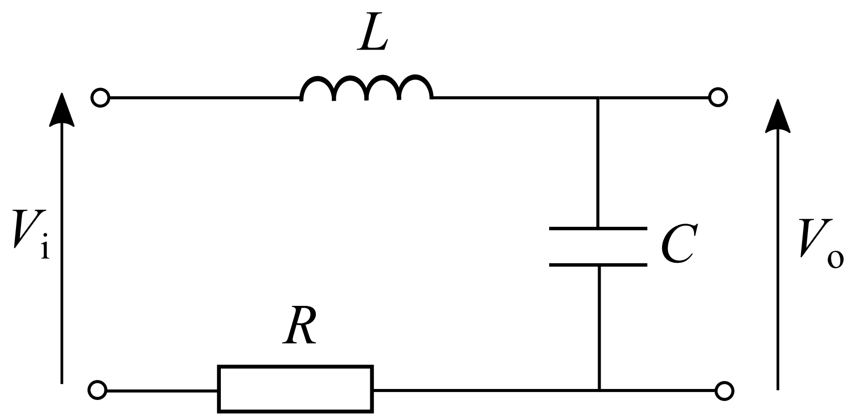
b) Show that the impedance of a resistor R in parallel with a capacitor C is $R/(1+\mathrm{i}\omega RC)$.

The figure below shows the circuit diagram of an input voltage V_1 being applied to the input terminals of an oscilloscope through a cable and a compensating probe. The input impedance of the oscilloscope itself is equivalent to a resistance of 1 MΩ in parallel with a capacitance of 30 pF, between its input terminal and ground. The cable to the probe contributes an additional capacitance of 45 pF between the oscilloscope input and ground, and the probe itself consists of a 9 MΩ resistance in parallel with a capacitor C_1.

Determine the value of the capacitor C_1 which makes the ratio of V_2/V_1 independent of frequency. With this value of C_1, determine the input impedance seen at the input to the probe (at V_1), expressing your answer in the form of a resistor in parallel with a capacitor.



23. The series resonant circuit shown below is driven by a voltage V_i oscillating at angular frequency ω.



Obtain a general expression for the voltage V_o. Find the amplitude of V_o and its phase relative to V_i for $\omega = 0$, $\omega = \omega_0 = 1/\sqrt{LC}$ and $\omega = \infty$; hence sketch how the amplitude and phase of $V_\mathrm{o}/V_\mathrm{i}$ vary as a function of frequency. (You may assume that the resistance R is very low, such that $R \ll 1/(\omega_0 C)$ – under these circumstances the maximum value of $|V_\mathrm{o}/V_\mathrm{i}|$ occurs when $\omega \approx \omega_0$.)

The circuit is to be used as a filter to pass frequencies within ± 20 kHz of 10 MHz. If the inductance L is chosen to be 100 μH what values of C and R should be chosen?

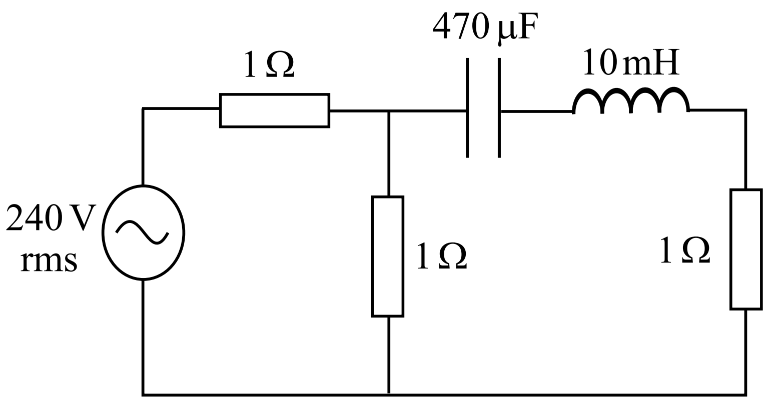
24. For the circuit below find:

(a) the rms current drawn from the 240 V rms supply if the frequency is 50 Hz, and the phase of the current relative to the applied voltage;

(b) the frequency which maximises the current drawn from the 240 V rms supply, and the value of this current.

(c) Find the voltage across the capacitor at that frequency.

(d) Explain why the voltage across the capacitor is greater than the total voltage applied to the circuit.



Answers to problems

1. $0.628\,\mathrm{m\,s^{-1}}$.
2. (c) $T/\sqrt{2}$; (d) $T/2$; (e) $1.6\,\mathrm{m\,s^{-2}}$.
3. (d) 0.2 %.
4. $0.150\,\mathrm{kg\,m^2}$.
5. (a) $\omega^2 = 2k/M$; (b) $\omega^2 = k/M$.
6. $1.9 \times 10^3\,\mathrm{N\,m^{-1}}$, $6.5 \times 10^{13}\,\mathrm{Hz}$.
7. (b) $t = (2n+1)/10\,\mathrm{s}$, where n is an integer.
9. $A = 0.028\,\mathrm{m}$, $\phi = -\pi/4$, $B_1 = 0.020$, $B_2 = 0.020$, $C = (0.010 - 0.010\mathrm{i})\,\mathrm{m}$, $D = (0.020 - 0.020\mathrm{i})\,\mathrm{m}$.
10. (b) $\theta = a\mathrm{e}^{-\gamma t}\cos(\omega_1 t + \phi)$ where $\omega_1^2 = \omega_0^2 - \gamma^2$; (i) 3.13 $\mathrm{s^{-1}}$, (ii) 3.35 s; 10.5.
12. (a) 3.16 $\mathrm{s^{-1}}$, (b) 6.28 $\mathrm{s^{-1}}$, (c) 5.09 cm
13. $A = \dfrac{F}{(k - m\omega^2)}$.
14. $A(\omega) = \dfrac{k\,a_S}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}$ and $\Phi(\omega) = -\tan^{-1}\left(\dfrac{b\,\omega}{k - m\,\omega^2}\right)$.
16. (c) $q = C\mathcal{E}\left(1 - \mathrm{e}^{-t/RC}\right)$.
17. (b) $\omega = 4.5 \times 10^3\ \mathrm{s^{-1}}$, $2.2 \times 10^{-6}\sin\omega t$ C, $0.01\cos\omega t$ A;
    (c) 5×10^{-7} J.
18. (b) phase = 120.3° at $t = 0.007\,\mathrm{s}$;
    (c) $Z_L = 3\,\Omega$ at 90°, $I_L = 3.3\,\mathrm{A}$ at −90°, $Z_C = 33.3\,\Omega$ at −90°, $I_C = 0.3\,\mathrm{A}$ at +90°;
    (d) $Z_{RL} = 10.4\,\Omega$ at +16.7°, $I_{RL} = 0.96\,\mathrm{A}$ at −16.7°,
    $Z_{LC} = 3.3\,\Omega$ at +90°, $I_{LC} = 3.0\,\mathrm{A}$ at −90°.
19. (b) $V_\mathrm{o} = 9.5\,\mathrm{V}$ at −18.2°;
    (c) $V_\mathrm{o} = 0$ for $\omega = 1000\,\mathrm{rad\,s^{-1}}$, $V_\mathrm{o} = 7.3\,\mathrm{V}$ at −43.2° for $\omega = 600\,\mathrm{rad\,s^{-1}}$, $V_\mathrm{o} = 7.0\,\mathrm{V}$ at +45.7° for $\omega = 1600\,\mathrm{rad\,s^{-1}}$.
20. (c) $10\,\Omega$, 90 mH.
22. (a) $R^2 = \frac{L}{C}$, (b) 8.3 pF, 10M ∥ 7.5pF.
23. 2.5 pF, 25 Ω.
24. (a) 127A $\angle 6.4°$, (b) 73.4 Hz, 160A, (c) 369V $\angle -90°$

Related IA Physics Tripos questions

Simple harmonic motion:

2013 C10
2014 C11
2015 A1
2015 C10
2016 C11
2017 C10
2017 C11
2018 C10 (last part)
2019 B7
2020 B7
2021 A3
2021 B8
2022 A3
2023 B8

Electrical circuits, Complex impedance and Oscillations:

2013 C11
2014 C10
2015 C11
2016 A3
2016 C10
2017 A6
2018 C11
2019 A2
2020 A1
2022 B8
2023 A2
2023 A5