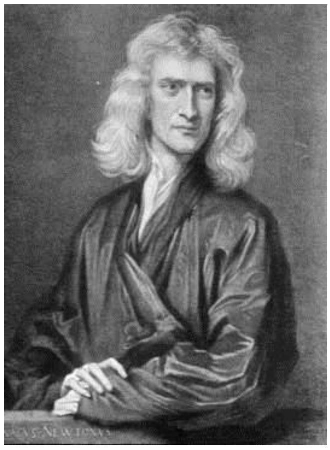
Rotational Mechanics & Special Relativity

Dr Sarah Williams

Examples Book 2023

$$\boxed{\underline{F} = m\frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2}}$$



Newton

Einstein

$$\boxed{E = \gamma mc^2}$$

ROTATIONAL MECHANICS and SPECIAL RELATIVITY

Dr S Williams

Mechanics in Rotational Motion: *Centre of mass*: calculation for a solid body by integration. *Turning moments***:** lever balance; conditions for static equilibrium. *Circular motion*: angle, angular speed, angular acceleration; as vectors. *Moments as vectors*: vector turning moment; vector moment of a couple. *Moment of inertia*: calculation of moment of inertia; theorems of parallel and perpendicular axes. *Angular momentum*: concept and definition; conservation; angular impulse; rotational kinetic energy. *Rotational oscillations*: the physical pendulum. *General motion of a rigid body*: example of solid cylinder rolling down a plane. *Rotating frames*: centripetal force. *Equivalents*: linear and rotational quantities. *Gyroscope*: how it works; precession.

Special Relativity: *Frames of reference*: general ideas**.** *Historical development*: problems with classical ideas; the Aether; Michelson-Morley experiment. *Inertial frames*: Galilean transformation. *Einstein's postulates*: statement; events, and intervals between them; consequences for time intervals and lengths; Lorentz transformation of intervals; simultaneity; proper time; twin paradox; causality; world lines and space–time diagrams. *Velocities*: addition; aberration of light; Doppler Effect. *Relativistic mechanics*: momentum and energy; definitions; what is conserved; energy– momentum invariant. Nuclear binding energies, fission and fusion.

ISAAC PHYSICS – join the lecture group:
Isaac Physics will be used as an integral part of this lecture course and therefore all students should take the following 5 steps:

1. Login / sign-up for a FREE account on Isaac (make sure you will remember your password or use Google or Facebook – Isaac will NOT post to either of these)
2. Click on this link https://isaacphysics.org/account?authToken=V7BPWF
3. Click OK on the pop-up that asks you to agree to share your data with me ("your data" means that I am able to see your progress with the questions that I have set.)
4. Go to https://isaacphysics.org/account and click on the tab "Beta Features". Tick the check box next to "Equation Text Entry".
5. Go to your assignments where you will see the first 2 questions for preparation for the first lecture. Assignments will be added after each lecture with typically 2 questions in each.

BOOKS & RESOURCES

isaacphysics.org: Questions in mechanics section – topics include statics, dynamics, circular motion, angular motion, SHM.

Understanding Physics, Mansfield & O'Sullivan (Praxis 2008)

Physics for Scientists and Engineers (with Modern Physics), Tipler P A & Mosca G (6th Edition Freeman 2008)

Planning your time:

The University's recommendation for the number of hours that students should spend studying per week is **45 hours**. In IA NST there are ~ 25 hours contact time which leaves 20 hours personal study time to be shared across the 3 experimental subjects and mathematics (i.e. ~5 hours per subject).

Developing confidence in problem-solving in physics takes practice, and access to problems of varying difficulty. I have therefore designed this course to keep within the recommended workload BUT maximise your exposure to **problems** which **are** our **key** to understanding physical concepts.

You should aim to:

a) spend a maximum of 40 mins preparing for each physics lecture (total of 2 hours per week).

   This will involve reading through the lecture notes and attempting the Isaac Physics examples that have been set for lecture preparation and for lecture revision.

   It is important that all students do this and answer the questions **online through Isaac**. Isaac provides you with hints and feedback, and me with a summary that will enable me to see if there are any particular questions that have caused issues so that I can go through them in my weekly online surgery (tutorial) - see below.

b) spend a maximum of 3 hours answering questions for supervisions –some questions for each supervision will have hints and support on Isaac Physics to save you a little time and provide you extra support – answers will not be given in the handout but Isaac will mark them instantly and give you feedback if you have made a common mistake.

   If you are able to complete the standard 6 questions in less than 3 hours additional problems are provided for you to fill your time.

Additional help and support – online office hours:

To help you to make the most of your personal study time in physics I will be providing online office hours (Details on Moodle) where I will go through:

- how to solve the preparation and review questions that have caused the class the most consternation (as determined by the class progress on Isaac).
- selected Tripos questions that summarise the topics / concepts covered in lectures each week
- any other questions that people have – I will help with supervision questions but only AFTER you have been through them with your supervisors first.
- It will be up to you to ask questions too – I will not just go through all the questions from the week.

About the examples / problems for this course.

So for problem solving practice I will provide:

i. Questions for preparation for each lecture and to practice the lecture content. These questions are designed to take a maximum of 20 mins each and are revision either of material that you have done at school or that has just been taught in the lecture. If after 20 mins you haven't solved the problem, stop and visit the online office hours.

ii. 24 examples which are contained in this booklet. The answers are given here for all the questions **except** those that are on Isaac. This is because Isaac will give you direct feedback as to whether your answer is right AND give you hints and videos to help you on your way if you are finding it difficult to get in to the problem. If you make a common mistake Isaac will also give you specific feedback too.

You can, of course, do as many questions as you like but it is sufficient to do the 24 standard examples written out in this book, by the beginning of Lent Term.

iii. Additional examples at the end of each section (i.e. RM and SR - pages have a grey edge), plus Tripos questions for revision.

The 24 standard problems have been carefully chosen to illustrate most of the material of the course and to aid your understanding. **To keep up you will need to attempt about six per week and I indicate when questions can be done in the lecture handout**, but be guided by your supervisor. Keep all your written work and plan your solutions, don't just write down random equations or numbers. This will help you with revision in the Easter Term.

Alternatively, if you want to further challenge your understanding, you and your supervisor may discuss substituting some of the standard questions with additional questions or may use the additional questions to discuss during your supervisions.

I also suggest further examples taken from past Tripos papers as good revision questions and for those that want to try out the exam style.

You may find some of these examples are more difficult than those you have met before. Don't be alarmed if it takes you some time to learn the techniques required to answer them. Talking to others, your colleagues and in particular your supervisor, is a valuable way to explore different approaches to answering problems. Remember that you are not in competition with your colleagues, it is often very helpful, and quite satisfactory, to work with another student such as your supervision partner.

There follows a set of guidelines which you should adopt when you tackle **any** physics problem - **NEVER** be tempted to skip step 1 (diagram). Please use them – they really will help! Particular advice for answering relativity problems is provided in the Special Relativity section.

Additional online examples from Isaac Physics.
Isaac Physics can provide you with as many (or as few) questions as you wish to practise, at a level that is most helpful to each of you individually (either extension materials or confidence building questions).

Of the 6 topics available within the mechanics section, those most closely aligned to this course are those of statics, dynamics, kinematics and circular and angular motion. Some questions in the SHM section you may also find useful. You can select your own questions by topic and level here https://isaacphysics.org/gameboards/new .

How to solve Physics problems

Here is some general advice which you should apply to *every* Physics problem you tackle. Now is the time to develop the techniques which make the problems easier to understand and which can provide you with a framework within which to think about Physics.

1. **Draw a *diagram***
   A diagram always helps to clarify your thoughts. **Use it to define the symbols you need to use** (see 3 below). Make it big enough and be tidy.

2. **Think about the *Physics***
   Ask yourself what is going on, and write it down in words in just one or perhaps two sentences. Try to *understand* the problem qualitatively before writing down any equations. Do not just write down equations!

3. **Stay in *symbols* until the end**
   At school you may have been taught to make calculations numerically rather than algebraically. However, you usually give yourself a big advantage if you delay substitution of numerical values until the last line as it enables you to check dimensions at every stage, and quantities often cancel before the last line. An exception to this rule arises where some terms are dimensionless factors which are simple fractions.

4. ***Check* the dimensions**
   Think about the dimensions of every quantity even as you write it down. You will find this a discipline which helps enormously to avoid errors and helps understanding. Make sure that the dimensions of your final equation match on each side before you make a numerical substitution. Write down the units of your answer at the end e.g. 4.97 J kg^{-1}.

5. **Does the answer *make sense*?**
   You will probably have an idea of what looks about right, and what is clearly wrong. Many mistakes are simple arithmetic errors involving powers of ten. If in doubt, check your substitutions.

Lecture preparation problem list (also shown in the lecture handout).

The level of difficulty for each question is given in brackets (L1 is the lowest level, L6 is the hardest).

Lecture **1 preparation** (to be done before lecture 1):

- Symmetry and Centre of Mass (*L1*)
- Shelf and Brackets (*L3*)

Lecture **1 review** (to be done before lecture 2):

- Weight of a Lorry (*L4*)
- Space Monster Attack (*L5*)

Lecture **2 review** (to be done before lecture 3):

- A Fairground Ride (*L4*)
- Moments of Inertia (*L5*)

Lecture **3 review** (to be done before lecture 4):

- An Audio CD (*L4*)
- Old Fashioned Record (*L5*)

Lecture **4 review** (to be done before lecture 5):

- Sphere Versus Cylinder (*L4*)
- T Shaped Pendulum (*L6*)

Lecture **5 review** (to be done before lecture 6):

- Ping-Pong on a Bus (*L4*)
- Space Justice (*L6*)

Lecture **6 review** (to be done before lecture 7):

- The Michelson-Morley Experiment (*L5*)
- Light Clock (*L6*)

Lecture **7 review** (to be done before lecture 8):

- A Lifeboat (*L4*)
- Angle between Two Identical Masses (*L6*)

Lecture **10 review** (to be done before lecture 11):

- Maximum Deflection of a Particle (*L6*)

Standard problems.

Answers are given to all problems that are not on Isaac Physics. To answer the questions on Isaac, login and go to "my assignments" from the menu bar or use the link given in the question. You can check your answer on Isaac Physics but also find hints and videos to help with the problems.

A question about calculating the position of the centre of mass etc. by integration.

1. (a) A uniform solid cone has a height b and a base radius a. It stands on a horizontal table.

(i) Draw a diagram showing the cone divided into thin horizontal discs, each of thickness $\delta\mathrm{h}$. Find an expression for the volume of the disc at height h above the base. Integrate over all the discs to find the total volume V in terms of a and b.

(ii) The height, b_0 , of the centre of mass is defined by the relation

$$b_0 = \sum_i \frac{m_i h_i}{M}$$

where M is the total mass, m_i is the mass of the disc at height h_i and the sum is taken over all the discs. Treat the sum as an integral, and hence find b_0 in terms of b.

(b) A uniform solid cylinder of radius r and length l is cut into two equal parts along its cylindrical axis. Find the position of the centre of mass of either part.

{ (a) i. $V = \frac{\pi}{3}\mathrm{a}^2\mathrm{b}$; ii. $b_0 = b/4$ (b) $4r/3\pi$ }

A question about extended bodies in static equilibrium.

2. Two rough cylinders, each of mass m and radius r, are placed parallel to each other on a rough surface with their curved sides touching. A third equivalent cylinder is balanced on top of these two, with its axis running in the same direction.

For this setup to be stable, what is the minimum coefficient of friction between the horizontal surface and the cylinders?

{https://isaacphysics.org/questions/nst1A_RM_q2}

A question using Newton's Second Law for rotational mechanics to calculate the angular acceleration of the body. For extension and practice, derive the moment of inertia of the sphere.

3. A uniform, solid sphere of density ρ and radius a, is rolling down a perfectly frictional plane inclined at angle α to the horizontal. The moment of inertia of the sphere, $I = \frac{8}{15}\pi\rho\mathrm{a}^5$.

(a) Use the rotational form of Newton's Second Law to find an expression for the angular acceleration of the sphere as it rolls down the plane in terms of ρ, a and the frictional force F that acts on the sphere.

(b) By integrating over a series of thin discs, discs from $-a$ to a, show that the moment of inertia of the solid sphere is indeed $\frac{8}{15}\pi\rho\mathrm{a}^5$.

{ (a) $15F/8\rho\pi a^4$ }

Break this question down into parts: first draw a diagram, and then find the frictional force on an element of the disc … etc.

4. A uniform disc of mass m and radius a, rotating at an angular speed of ω, is placed flat on a horizontal flat surface. If the coefficient of friction is μ, find the frictional torque on the disc, and hence calculate the time it takes to come to rest.

{https://isaacphysics.org/questions/nst1A_RM_q4}

Practice using the parallel and perpendicular axis theorems to find moments of inertia.

5. State the parallel and perpendicular axes theorems.

(a) Calculate the moment of inertia of a uniform square plate of side a and mass m about an axis through its centre and parallel to a side.

(b) Use the perpendicular axis theorem to find the moment of inertia through the centre and perpendicular to its plane.

(c) More generally, show that the moment of inertia of a square plate about *any* axis in its plane through its centre is the same.

(d) Use the theorems of parallel and perpendicular axes to find the moment of inertia of a hollow thin cubical box of side a and total mass M about an axis passing through the centres of two opposite faces.

{ (a)$ma^2/12$; (b)$ma^2/6$; (d) $5Ma^2/18$ }

Here is an example of an "angular collision". Note that angular momentum is conserved only if the system is isolated.

6. Two gear wheels are cut from the same uniform sheet of metal, the mass per unit area of which is σ. One gear wheel has radius a and the other radius $2a$ with twice as many teeth. They are mounted on parallel light axles through their centres and perpendicular to their faces just far enough apart not to mesh.

(a) Calculate their moments of inertia in terms of σ and a.

(b) The larger wheel is now spun with angular speed ω. What is the total angular momentum of the system in terms of σ, a and ω ?

(c) The gear wheels are now suddenly meshed, but their axles remain in the same plane. During this process energy is lost, and the wheels exert equal and opposite tangential impulses $\int F(t)dt$ on each other. Consider the effect of the angular impulses $\int rF(t)dt$ associated with these forces on the angular momentum of each wheel, where *r* is the radius of the wheel concerned. Hence show that the angular speed of the larger wheel falls by 20%.

(d) Show that the total angular momentum falls by 30%. Why is it not conserved in your calculation?

{ (a) $\sigma\pi a^4/2, 8\sigma\pi a^4$ (b) $8\sigma\pi a^4\omega$ }

7. The hollow cubical box of question 5 (d) is suspended from a horizontal frictionless hinge along one of its edges. The box is displaced slightly from equilibrium. Show that when displaced through some small angle θ it obeys the following equation of motion

$$\ddot{\theta} = -\frac{9g\theta}{7\sqrt{2}a}$$

(As you'll see later in the oscillations course, this corresponds to simple harmonic motion with a period $T = 2\pi\sqrt{\frac{7\sqrt{2}a}{9g}}$ *)*

These are examples of "solid body dynamics" in which there is both rotational and linear motion

8. A thin, uniform bar of length l and mass M is suspended horizontally at rest. It is suddenly released, and at the same instant, is struck by a sharp blow vertically upwards at one end — the duration of the impulse can be taken to be negligibly short.

(a) "The centre of mass moves as if acted upon by the (vector) sum of the external forces". Describe the motion of the bar's centre of mass.

(b) "The rotation about the centre of mass is caused by the sum of the moments of the external forces". Describe the rotational motion of the bar.

(c) Hence, by combining your answers to (a) and (b), describe how the bar moves after being struck. Be as quantitative as possible, considering in particular

(i) the time taken by the centre of mass to reach maximum height, and

(ii) the rate of rotation.

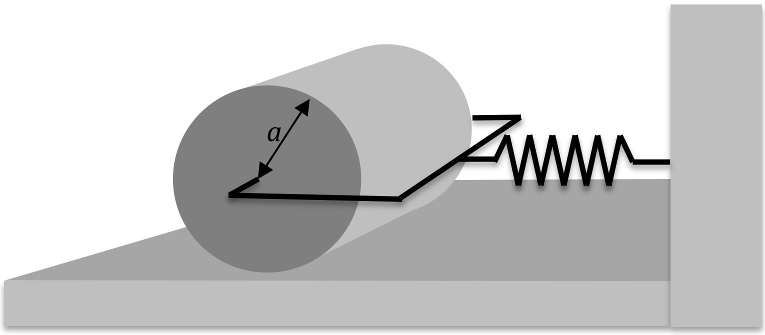
(d) In a particular experiment, the bar passes through its original position in the same orientation after a time, t. Demonstrate that, $t^2 = 2\pi nl/3g$, where n is an integer and g is the magnitude of the acceleration due to gravity.

{ (c) i. $l\omega/6g$; ii. $\omega = 6v_0/l$ }

9. A solid cylinder of mass M and radius a is free to roll (without slipping) on a horizontal surface and is connected to a light spring of constant k as shown in the diagram. The system is displaced from equilibrium by rolling the cylinder so that the spring extends a small amount along its axis. Show that the displacement x obeys the following equation of motion

$$\ddot{x} = -\frac{2kx}{3M}$$

(Similar to question 7, this is simple harmonic motion but this time with $T = 2\pi\sqrt{\frac{3M}{2k}}$*)*



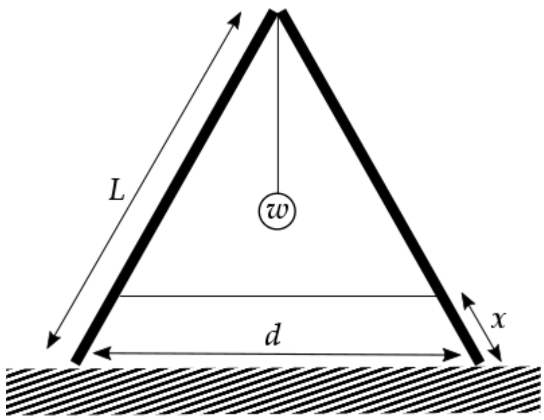
These are examples of gyroscopic motion with constant precession.

10. A gyroscope wheel is at one end of an axle of length d. The other end of the axle is suspended from a string of length s, that makes a fixed angle θ with the vertical. The wheel is set into motion so that it executes uniform precession in the horizontal plane. The wheel has mass M and moment of inertia about its centre of mass I. Its angular speed about the axle through its centre is ω. Neglect the mass of the axle and the mass of the string and assume that $|\omega| \gg |\Omega|$. What is the direction and magnitude of the angular velocity of precession?

{https://isaacphysics.org/questions/nst1A_RM_q10}

Additional Problems [not compulsory]

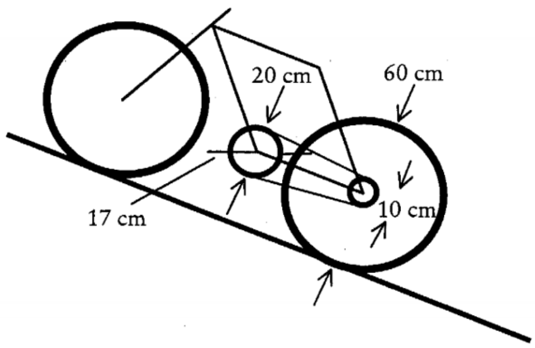
A1. A truss is made by hinging together two uniform rafters each of weight W and length L. They rest on an essentially frictionless floor, so that their ends are a distance d apart and are held together by a tie rope attached at points a distance x from the ends. A load of weight w is held at their apex. Find the tension in the tie rope.



$\{ \frac{d}{2\sqrt{(4L^2-d^2)}} \frac{(W+w)}{\left(1-\frac{x}{L}\right)} \}$

A2. A cyclist, riding up a steep hill, finds that it is just possible to remain stationary by standing on the forward pedal when the pedal arm is horizontal. The arm of the pedal has a length of 17 cm , the front gear has a diameter of 20cm , the back gear has a diameter of 10cm , and the wheel has a diameter of 60cm. If you ignore the mass of the bicycle, what is the gradient of the hill?

{ ~16.5° }



A3. A uniform narrow rod, 1m in length, has the section from 60cm to 90cm from one end replaced by a mass-less section of the same length.

(a) Calculate by integration the new location of the centre of mass of the rod.

(b) Show that the same result can be obtained by calculating the centre of mass of the centres of mass of the two remaining sections.

(c) Show that the same result can be obtained by calculating the centre of mass of the original uniform rod and that of the 30cm section if it is treated as having negative mass.

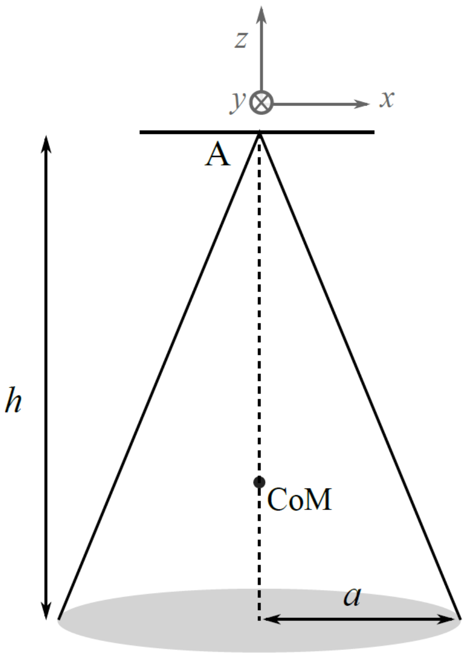
(d) A uniform sphere of radius r has a sphere of radius $r/2$ cut out of it. The cut-out section just touches the larger sphere's centre. Where is the centre of mass of this damaged sphere?

{ (a) at 39.3cm from one end; (d) $r/14$ from the centre of the large sphere, on the axis of symmetry of the damaged sphere, away from the missing small sphere}

A4. A clockmaker wishes to design a new clock with a timing mechanism that uses the period of oscillation of a solid, uniform cone of mass M. A cone is suspended from its apex, A, so that it rotates about the x-axis such that its centre of mass can swing freely in the $y-z$ plane, as shown in the diagram.

Find an expression for the moment of inertia of the cone about the x-axis at A in terms of the mass of the cone M, its radius a and its height h.

{$\frac{3M}{20}(a^2+4h^2)$ }



The following question is an example of the conservation of angular momentum, and the transformation of energy between different forms. Imagine that you are an astronaut in the spacecraft sitting on a bench in the middle and looking out of a window in the curved side of the spacecraft. You are holding the string of part (b). As you slowly let out the string, think about the tension you feel in the string, the work you do, and what you see out of the window.

A5. A spacecraft can be regarded as a uniform, thin, hollow cylinder, 1m in diameter and of mass 250kg (including a small mass, m=50 g). The flat ends of the cylinder are of negligible mass. It is spinning about its cylindrical axis with a period of 3 s.

(a) What is the angular momentum of this isolated system? This quantity remains constant for all time.

(b) The 50 g mass, m, is attached to a long string and slowly let out from the side of the cylinder until the period has increased to 10 minutes. Explain why the period increases and calculate the length of the string.

(c) The cylinder slows down when the string is let out, so it must feel a torque. Draw a diagram showing clearly how the torque arises.

(d) Calculate the initial and final kinetic energies of rotation of the system.

(e) An electric motor in the cylinder is now switched on to wind in the string slowly. Explain why it has to do work. Where does the work go?

{ (a) $130.9\ \mathrm{kg\,m^2s^{-1}}$; (b) 499 m; (d) $137.1\ \mathrm{J}, 0.69\ \mathrm{J}$ }

A6. A (i) solid cylinder, (ii) hollow cylinder, (iii) and solid sphere, each of radius a, roll down a ramp from height h_0 to perform a classic "loop the loop" stunt on meeting a circular track of radius R, in the vertical plane. *[Assume that the track is perfectly frictional, that the objects* ***just*** *make the stunt and that the join between ramp and loop is ideal]*

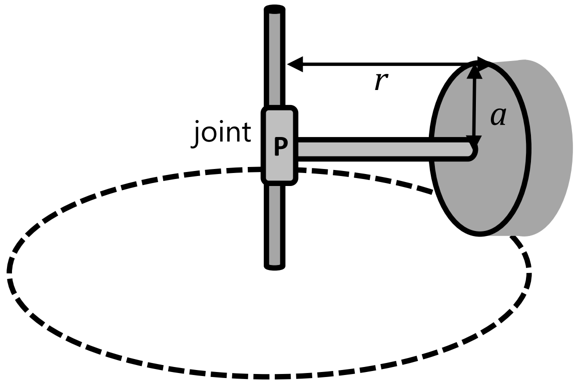
(a) Find expressions for the starting heights, h_0, in terms of R for each of the objects (i-iii) given that their radii $a \ll R$.

(b) What starting height must the sphere have if the track now has a V-groove cross section with total internal angle of 60 degrees?

{ (a) (i-iii) $h_0 = \frac{R}{2}\left(5 + \frac{I}{ma^2}\right)$; (b) $h_0 = \frac{33R}{10}$ }

A7. A thin uniform rod of mass M is supported horizontally on knife edges at each end. If one of the supports is suddenly removed, show that the force on the other end is instantaneously reduced from $Mg/2$ to $Mg/4$, where g is the magnitude of the gravitational acceleration.

A8. In a mill, grain is ground by a massive wheel that rolls without slipping in a circle on a flat horizontal mill stone driven by a vertical shaft. The rolling wheel has mass M, radius a and is constrained to roll in a horizontal circle of radius$r$, at angular speed Ω. The wheel pushes down on the lower mill stone with a force equal to twice its weight (normal force). The mass of the axle of the wheel can be neglected. What is the value of Ω. ? *[Assume that both the angular speed of the wheel and Ω remain constant]*



$\{\Omega = \sqrt{\frac{2g}{a}}.\ \}$

Several further questions on rotational mechanics from past Tripos papers (not compulsory):

2009 B7: {Answer: $\omega = 6\sqrt{2gh}/7l$; $h/49$}

2007 B9: {Answer: mu, applied $2a/5$ above centre; $7mu^2/10$; $\sqrt{\frac{10}{7}g(r+a)(1-\cos\theta)}$ }

2004 A1: {Answer: $h_0/2$}

2004 A2: {Answer: $34ma^2/3$}

2003 B7: {Answer: $\sqrt{3g/L}$; $m\sqrt{gL/3}$; $m\sqrt{3gL/4}$ at $2L/3$ from the axis}

Relativistic Kinematics *is the application of the Special Theory of Relativity to space and time. Please read the following general advice on how to tackle kinematical relativity problems before proceeding.*

Some or all of the following 'rules' can be applied to solve any kinematical problem in Special Relativity. If you apply the rules carefully, without first muddling yourself with too much potentially confusing thought about what contracts and what dilates etc., you will get the right answer. You can ponder about what it all means when you know that you have the right answer!

(i) **Identify the events**. Label them A, B, C etc. Thus event A may be the flash of a light, B the spaceship exploding, C the arrival of a message at the Earth etc.

(ii) **Draw diagrams showing the events in the relevant frames of reference.** Thus you might show events A and B as seen both in the Earth frame and in the rocket frame.

(iii) **Write down the *intervals* between the events in all frames.** Set these equal to known quantities where you can and put a question mark where you can't. Thus you might write $\Delta x'_{AB} = l_0$; $\Delta x_{AB} = ?$; $\Delta t'_{AB} = \frac{l_0}{c}$; $\Delta t_{AB} = ?$

(iv) **Apply the Lorentz transformation to the intervals to find the unknown values**. If S′ is the frame moving at speed v parallel to, and in the direction of, the positive x axis of frame S, then the Lorentz transformation of the interval $(\Delta x, \Delta y, \Delta z, \Delta t)$, between two events as observed in the S frame, and the interval $(\Delta x', \Delta y', \Delta z', \Delta t')$, between the *same* two events as observed in the S′ frame is:

$$\begin{aligned}\Delta t' &= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)\\ \Delta x' &= \gamma(\Delta x - v\Delta t)\\ \Delta y' &= \Delta y\\ \Delta z' &= \Delta z\end{aligned}$$

$$\begin{aligned}\Delta t &= \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)\\ \Delta x &= \gamma(\Delta x' + v\Delta t')\\ \Delta y &= \Delta y'\\ \Delta z &= \Delta z'\end{aligned}$$

Where $\gamma = \sqrt{\frac{1}{\left(1-\frac{v^2}{c^2}\right)}}$

Or using 4-vectors and matrices:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Standard problems.

Special Relativity kinematics: these two examples are about reference frames. The first example provides practice in applying the concepts of length contraction and time dilation.

11. (a) A positive kaon (K^{+}) has a lifetime of 0.1237 μs in its own rest frame. It has a speed $0.990c$ relative to a laboratory frame of reference. How far can it travel in the laboratory frame during its lifetime according to classical physics and special relativity?

(b) Two events occur in the same place in the laboratory frame and are separated by a time interval of 2 s. The time interval between these two events when measured from a rocket's frame is 4 s.

(i) Deduce the Lorentz factor γ and hence calculate the speed of the rocket relative to the laboratory frame.

(ii) Calculate the distance between the two events in the rocket frame.

(c) A metre stick is 2 cm wide and is aligned in the north-south direction. How fast and which direction, relative to an observer, is the metre stick moving if its length appears the same as its width?

{ (a) 36.7 m, 260.4 m ; (b) (i) 2, $\sqrt{3}$c/2 (ii)$2\sqrt{3}c$; (c) $0.9998c$ }

The next example is a variation of the so-called twin paradox. There is no paradox if you are careful to analyse the circumstances of each twin strictly according to the rules of special relativity. A paradox only arises if your analysis is faulty, or your thinking is woolly!

12. A spaceship sets off from Earth for a distant destination, travelling in a straight line at a uniform speed of 3c/5. Ten years later, as measured on the Earth, a second spaceship sets off in the same direction with a speed of 4c/5. The captains of the two vessels are twins.

(a) For how long, in the Earth's frame, do each of the spaceships travel before the second spaceship catches up with the first?

Consider three events: (A) the slower spaceship leaves the Earth; (B) the faster spaceship leaves the Earth; and (C) the faster spaceship catches up with the slower spaceship.

(b) If the event A has coordinates $x = 0$, $t = 0$ in the Earth's frame, what are the coordinates of the other two events, also as observed in the Earth's frame?

(c) By transforming these events to frames moving with the slower and faster spaceships respectively, determine which of the twins is older, and by how much, when the faster spaceship catches up with the slower spaceship.

{ (a) 40 years, 30 years; (b) x=0, t=10 years and x=24 light years, t=40 years;
(c) the first captain is older by 4 years }

Two further examples of transformation of distances and times between different frames.

13. As a spaceship passes the Earth with a speed of 0.8 c, observers on this spaceship and on the Earth agree that the time is 12:00 in both places. Thirty minutes later, as measured on the spaceship's clock, the spaceship passes an interplanetary navigation station fixed relative to the Earth. The clock on the interplanetary navigation station reads Earth time.

(a) What is the time on the navigation station clock as the spaceship passes?

(b) How far from Earth, as measured in the Earth's frame, is the navigation station?

(c) As the spaceship passes the navigation station, it reports back to Earth by radio. When, according to a clock on the Earth, is the signal received?

(d) There is an immediate reply from Earth. When, according to the spaceship's clock, is the reply received at the spaceship?

{ (a) 12:50; (b) 40 light minutes =7.2×10^{11}m ; (c) 13:30; (d) 16:30}

14. A flash of light is emitted from the tail of a rocket of length l_0 (measured in its rest frame) towards the nose. The flash is reflected by a mirror at the nose and received back at the tail. If the rocket is moving at speed v relative to the Earth, what are the time intervals measured in the Earth's frame between the emission, reflection, and reception of the flash?

{$\sqrt{(c+v)/(c-v)}.(\mathrm{l_0/c})$; $\sqrt{(c-v)/(c+v)}.(\mathrm{l_0/c})$ }

An example of the effects of simultaneity as viewed in different frames

15. A very fast train, of length L_0 (measured in its own frame), rushes through a station which has a platform of length L (<L_0) in the rest frame of the station.

(a) What is the speed v of the train such that the back of the train is opposite one end of the platform at exactly the same instant as the front of the train is opposite the other end, according to observers on the platform?

(b) According to these observers, two porters standing at either end of the platform (a distance L apart) are foolish enough, but have quick enough reactions, to kick the train simultaneously as it passes, thereby making dents in it. When the train stops, the dents are found to be a distance L_0 apart. Explain in words and with diagrams how the difference between L and L_0 is explained by

(i) an observer on the platform, and

(ii) an observer travelling on the train?

(c) According to an observer on the train, what is the length of the platform, L', in terms of L_0 and γ.

{ (a)$v = c\sqrt{1-\left(\frac{L}{L_0}\right)^2}$; (c)$L' = L_0/\gamma^2$ }

The relativistic transformation of speeds

These are the formulas you must use when adding speeds together, or transforming speeds from one inertial frame to another. For example, consider the case of a passenger walking along the corridor of a carriage of a train in the same direction as the train is travelling from a classical, non-relativistic, point of view. Inside the train, the passenger's speed is u_x' relative to the carriage. If the train is also moving at speed *v* relative to the tracks, then the passenger's speed relative to the tracks is $u_x = u_x' + v$. This result applies when **both** u_x' and *v* are very much less than the speed of light, *c*,

i.e.: The classical approximation (if both $u_x' \ll c$ and $v \ll c$)

$$\begin{aligned} u_x &= u_x' + v \\ u_y &= u_y' \\ u_z &= u_z' \end{aligned}$$

Now consider the case in which the passenger is an astronaut moving along a spaceship at a speed u_x' relative to the spaceship in the same direction as the spaceship is travelling. Suppose that the spaceship's speed with respect to the Earth is v, and that v is comparable with c. In this case, you *must* use the relativistic transformation of speeds formulas to find the astronaut's speed, u_x, with respect to the Earth as follows:

Relativistic (*always true*)

$$\begin{aligned} u_x &= \frac{u_x' + v}{1 + u_x' v/c^2} \\ u_y &= \frac{u_y'}{\gamma(1 + u_x' v/c^2)} \\ u_z &= \frac{u_z'}{\gamma(1 + u_x' v/c^2)} \end{aligned}$$

Two examples of the way speeds transform between different frames

16. Two sub-atomic particles approach each other with velocities (relative to an observer at rest in the laboratory) of $3c/5$ and -2c/5. What is the speed of one particle as observed in the frame of reference of the other? With what velocity would the observer have to move to measure their velocities as equal and opposite?

$\{25c/31, 0.134c\ \}$

17. A π^0 meson, travelling with velocity $(3c/4,0,0,0)$ in the laboratory frame, S, decays into two photons in the xy- plane. In *S′*, the rest frame of the meson, one of the photons is emitted at an angle $\theta' = 60°$ to the x'-axis. In frame S, calculate the x and y components of the velocities of the two photons, and hence their angles of emission with respect to the x axis.

{https://isaacphysics.org/questions/nst1A_SR_q17}

A question about the relativistic Doppler Effect. One way of thinking about this is to imagine that the reflected pulses are coming from the image of the rocket as seen in a plane mirror (the planet's surface).

18. A rocket moving away from the Earth with speed v emits light pulses with a frequency f_0 of one pulse per second as measured by a clock on the rocket.

(a) Show that the rate, f, at which the pulses are received on the Earth is given by

$$f = f_0\sqrt{\frac{c - v}{c + v}}$$

(b) The rocket travels to a distant planet, and the signals are received on Earth both directly and by reflection from the planet. The pulse rates for the two signals are found to be in the ratio 1 : 2. Explain why this is so, and deduce the speed of the rocket.

(c) If the rocket transmits only during its flight and the number of pulses received directly is 10^4, what is the distance of the planet from the Earth?

{ (b) $c/3$; (c)1.06×10^{12}m }

Relativistic Dynamics *is the application of the Special Theory of Relativity to energy and momentum etc. Please read the following general advice on how to tackle relativistic dynamics problems before proceeding with the examples.*

It is not so easy to identify a fixed set of 'rules' for solving relativistic dynamics problems. However, there are certain principles which can be acknowledged, some or all of which may be helpful in any given case:

(i) **Identify the events**, and label them A, B, C etc.

(ii) **Draw diagrams** showing the conditions before and after the events in a given frame (e.g. the energies, momentums, masses, and velocities of particles before and after a collision event in the laboratory frame).

(iii) If possible, **identify another frame** (such as the *zero-momentum frame*) in which it is easier to make calculations. Draw diagrams showing conditions before and after events in this frame.

(iv) **Apply the principle of the conservation of linear momentum** in any single frame before and after any event, i.e.

$$\sum_i \boldsymbol{p}_i = \text{constant},$$

Where $\boldsymbol{p} = \gamma_v m\boldsymbol{v}$, m is the mass, $\boldsymbol{v}$ is the velocity in this frame, and $\gamma_v^2 = \frac{1}{(1-v^2/c^2)}$

The momentum can also be expressed in terms of γ_v as $p^2 = |\boldsymbol{p}|^2 = m^2c^2(\gamma_v^2 - 1)$.

(v) **Apply the energy-momentum invariant**:

$$E_1^2 - p_1^2c^2 = E_2^2 - p_2^2c^2 = \text{constant}$$

Where E_1 and $\boldsymbol{p}_1$ are the *total* energy and the *total* momentum in the system respectively, evaluated at time 1 or in frame 1 etc. By 'system' we mean either a single particle, or a group of particles, or the contents of a whole frame. Note that $E^2 - p^2c^2$ is the same before and after *any* event when calculated in *any* inertial frame, provided that you don't change the system. *E* is the *total* energy given by

$$E = \sum_i \gamma_i m_i c^2,$$

Where γ_i is the value of γ appropriate to the i^{th} particle whose mass is m_i. It must include particles which are stationary (i.e. $\gamma = 1$) as well as moving particles. Add up all the energies, then square the total to get E^2.

Note that the kinetic energy is **not** $mv^2/2$, nor is it $\gamma mv^2/2$. The kinetic energy, K, is simply the difference between the energy a particle has when it is moving and the energy it has at rest, i.e.

$$K = \gamma mc^2 - mc^2 = (\gamma\text{-}1)\, mc^2.$$

Similarly, add up all the contributions to the momentum (as vectors) before squaring the magnitude of the result. **For a single particle:** $E^2 - p^2c^2 = m^2c^4$.

(vi) **Apply the Lorentz transformation to energy and momentum**. You will hear mention of *four-vectors* which transform according to the Lorentz transformation. There are many four vectors, such as the four vector (ct, x, y, z). Energy and momentum can also be combined to form the energy-momentum four-vector (E/c, p_x, p_y, p_z). If a particle has an energy *E* and a momentum given by $\underline{\boldsymbol{p}}$= ($p_x$, p_y, p_z) as observed in frame *S*, then the corresponding quantities in frame *S′* are given by

$$E'/c = \gamma\left(E/c - \frac{vp_x}{c}\right) \qquad E/c = \gamma\left(\frac{E'}{c} + \frac{vp'_x}{c}\right)$$
$$p'_x = \gamma\left(p_x - \frac{vE}{c^2}\right) \qquad p_x = \gamma\left(p'_x + \frac{vE'}{c^2}\right)$$
$$p'_y = p_y \qquad p_y = p'_y$$
$$p'_z = p_z \qquad p_z = p'_z$$

Where $\gamma = \sqrt{\frac{1}{\left(1-\frac{v^2}{c^2}\right)}}$

Or using 4-vectors and matrices:

$$\begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

The next few questions are about Relativistic Dynamics – i.e. how energy and momentum transform between moving frames. Please first read the advice on Pages 19 & 20.

19. A particle as observed in a certain reference frame has a total energy of 5 GeV and momentum of 3 GeV/c

(a) What is the velocity of the particle in this frame and therefore what is the value of γ?

(b) What is its mass in units of GeV/c^2?

(c) Observed in another frame of reference, the particle has a total energy of $4\sqrt{2}$ GeV and a momentum of 4 GeV/c. Use the formulae for the relativistic transformation of speeds, or the energy- momentum transformation, to find the relative speed of the two frames of reference, if the particles are moving in the same direction.

{ (a)$3c/5$; $5/4$;(b) $4\ \mathrm{GeV}/c^2$ (c) -0.186c }

20. Estimate the energy required to accelerate an electron from rest to half the speed of light. How does this compare with the result of the same calculation carried out non-relativistically? What is the ratio of the two energies? Is it important to consider relativistic equations at speeds = $c/2$

{79 keV; 64 keV; 1.24 }

21. If K represents the relativistic kinetic energy of a particle of mass $\boxed{m}$, show that

$$K^2 + 2mKc^2 = p^2c^2,$$

where p represents the momentum of the particle.

A particle of mass m and kinetic energy K strikes and combines with a stationary particle of mass $2m$, producing a single composite particle of mass $\sqrt{17}m$. Find the value of K

{$2mc^2$}

22. A particle of mass M disintegrates while at rest into two parts having masses of $M/2$ and $M/4$. Find the relativistic kinetic energies of each part.

[*Hint:* Use the result from the first part of question 21.]

{$3Mc^2/32, 5Mc^2/32$ }

Another collision question. Try transforming into the ZMF to simplify the problem. The value of the E-p invariant is the same before and after a collision, as viewed in any inertial frame.

23. A high-energy proton hits a stationary proton and produces a neutral π^0meson (a pion) with mass of 0.144 times the proton mass via the reaction

$$p + p \rightarrow p + p + \pi^0 \text{ meson}$$

(a) If the incident proton has just enough energy to allow this reaction to occur, what is the velocity (speed v and direction) of the final protons and the pion in the laboratory frame of reference?

(b) The neutral pion is observed to decay into two photons of equal energy in the laboratory frame. If θ is the angle between the π^0 direction and the direction of either of the photons (as observed in the laboratory frame), find an expression for $\cos\theta$ in terms of v and c and hence determine the opening angle between the two photons.

(c) If the π^0meson decays with a proper lifetime of 8.4×10^{-17}s (as measured in its rest frame), how far on average does it travel in the laboratory frame before decaying?

{https://isaacphysics.org/questions/nst1A_SR_q23}

24. An α-particle has twice the charge and four times the mass of a proton. If an accelerator imparts $\sqrt{6}$ times as much momentum to an initially stationary α-particle as it does to an initially-stationary proton, what is the accelerating voltage?

{ 1.89 GV }

Additional Problems [not compulsory]

A9. The space and time coordinates of two events as measured in a Galilean frame G are as follows:

$$\text{Event A: } x_A = x_0, \quad t_A = x_0/c$$
$$\text{Event B: } x_B = 2x_0, t_B = x_0/(2c)$$

Where $y = z = 0$ for both events.

(a) What is the speed and direction of travel of another Galilean frame G′ in which both events occur at the same place? (Use the Galilean transformation – i.e. the classical, non-relativistic transformation. Assume the "standard configuration" for the frames G and G′, with G′ moving at speed v along the $+x$- axis, and the origins of G and G′ coinciding at $x = x' = 0, t = t' = 0$)

(b) Comment on the result. In particular, explain why the Galilean transformation is inappropriate in this example.

(c) Can the events be seen to occur at the same time in any Galilean frame?

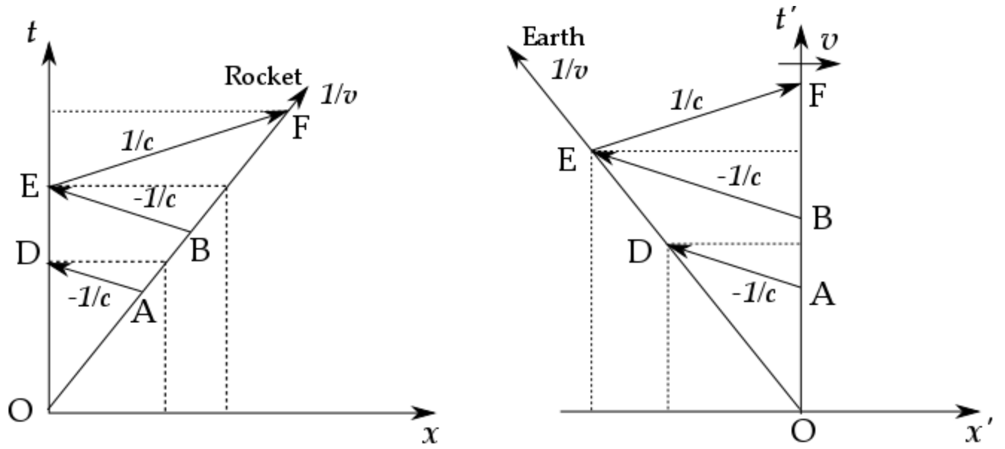
(d) Now use the Lorentz transformation to calculate the speed and direction of travel of an inertial frame S′ in which both events occur at the same time. (Take the "standard configuration" for the frames S and S′, which is that S′ moves at speed *v* along the *x*-axis, and the origins of S and S′ coincide at the instant $x = x' = 0, t = t' = 0$.)

(e) Comment on the result. In particular, explain why the Lorentz transformation is appropriate in this example.

(f) When and where do the events occur as measured in *S*′?

{ (a) $-2c$; (d)$-c/2$; (f) $t' = \sqrt{3}(x_0/c), x'_A = \sqrt{3}x_0, x'_B = 3\sqrt{3}x_0/2$ }

A10. A rocket leaves Earth travelling at a speed of *v*. After the rocket has travelled a distance it emits light signals towards Earth to communicate its position. After the second rocket signal is received on Earth a reply is sent from Earth to the rocket. The time-distance graphs below illustrate the Earth and Rocket frames.



(a) From these diagrams we see that $\Delta x'_{AB} = 0$, show therefore that the time between the two pulses leaving the rocket in the rocket's frame ($\Delta t'_{AB}$) is dilated compared with the Earth's frame (Δt_{AB}).

(b) From these diagrams we see that $\Delta x_{DE} = 0$, show therefore that the time between the two pulses received on Earth in the Earth's frame (Δt_{DE}) is dilated compared with the rocket's frame ($\Delta t'_{DE}$). Explain why the answer to (b) is the reverse of the answer to (a).

(c) Calculate how far the rocket has moved away from the Earth, in both frames, in the time between the two pulses being received on Earth (Δt_{DE} and $\Delta t'_{DE}$). What is the ratio between these two distances?

(d) Calculate the distance of the rocket from Earth in both frames (d and d') when the reply from Earth is received. Give your answer in terms of $v, \Delta t_{OF}$ and γ.

{ (a)$\Delta t'_{AB} = \frac{\Delta t_{AB}}{\gamma}$; (b))$\Delta t_{DE} = \frac{\Delta t'_{DE}}{\gamma}$; (c)$\frac{1}{\gamma}$; (d) $d = v\Delta t_{OF}, d' = \frac{v\Delta t_{OF}}{\gamma}$ }

A11. A rocket of length L_0 measured in its rest frame S′ is travelling away from an observer on Earth (frame S) with a velocity $u = 9c/41$. A light pulse is emitted from the nose of the rocket ($x' = \mathrm{L}_0$, $t' = 0$) and travels to the tail ($x' = 0$) - where it is reflected back to the nose and continues back and forth between the tail and the nose.

(a) Draw time-distance diagrams for the Earth (S) and rocket (S′) frames showing clearly the light pulse and its reflection and the rocket and Earth as appropriate. [Label **ALL** the events on your diagram and refer to the intervals in space and time using these labels as subscripts.]

(b) In S′, when does the light pulse reach the tail and when does it get back to the nose?

(c) How long does the light pulse take to travel from nose to tail and back again as determined by the observer on Earth (S)?

(d) By considering the passage of the light from the nose to the tail, calculate L(the length of the rocket in S) in terms of L_0 and hence show that the rocket is length contracted according to the observer in the Earth's frame. Show that the answer is the same if you consider the pulse from tail to nose.

{ (b)$^{L_0}/_c$ $^{2L_0}/_c$; (c) $^{4L_0}/_{5c}$ $^{5L_0}/_{4c}$; (d) $^{40L_0}/_{41}$ }

The following question can be done in many ways but it is intended as practice in the relativistic transformation of speeds.

A12. Angela is living on Mars and has decided that she would like to meet up with her friend Bob (who lives on Earth) at some location between them. Angela agrees that she will set off from Mars at 12 noon (in all frames) travelling at $u = -5c/13$ and at that instant she will send a message to Bob so that when he receives her signal he can then set off to meet her at velocity $v = 12c/13$. At the instant that Bob leaves Earth he sends a signal to Angela to say he has left. Angela replies with her location, as does Bob when he receives her second message. They continue to communicate back and forth until they meet.

(a) Draw time-distance diagrams for the Earth's frame (S), Angela's frame (S′) and a frame that is moving at $12c/13$ (S″). Include on all of them the passage of the signals between Angela and Bob.

[You will need to calculate Bob's velocity in Angela's frame (S′) and Angela's velocity in S″ and the Earth's velocity in both of these frames]

(b) At what time in S, S′ and S″ does Bob leave to meet Angela? [Mars is ~ 12 light minutes from Earth in S]

(c) How far apart in S, S′ and S″ are Angela and Bob when Angela receives Bob's message that he has left? Give you answers in units of light minutes.

(d) At what time do they meet in S, S′ and S″ and how far away from Earth are they in S, S′ and S″? Again give your answer in units of light minutes

{ (b) 12:12, 12:08, 13:00; (c) 16c/39, 64c/229, 16c/15; (d) 12:17.6, 12:16.3, 13:02.2; 5.21c 4.81c, 2.00c }

The next few questions are about relativistic mass and energy calculations.

A13. The bulk of the energy production in the Sun is from the conversion of hydrogen into helium via the proton-proton cycle of reactions. Overall, these reactions amount to the formation of one helium nucleus from four protons with the release of two massless neutrinos (each of average energy 0.25 MeV) and several photons: $4p^+ + 2e^- \rightarrow {}^4_2\mathrm{He}^{2+} + 2\nu + 6\gamma$

The kinetic energy carried off by the photons and alpha-particles is dissipated as heat which is the source of energy for the Sun's luminosity. The neutrinos escape from the Sun without interacting.

(a) Calculate the energy (in MeV) released by the formation of one helium nucleus from four protons used to heat the Sun. (The masses of a proton and of a helium nucleus are $1.00728\ m_u$ and $4.00150\ m_u$ respectively and $1m_u = 931.5\ \mathrm{MeV}/c^2$.)

(b) The total optical and infrared luminosity of the Sun is 3.9× 10^{26} W. Estimate the mass of hydrogen being converted to helium in the core of the Sun every second.

(c) What fractional rate of loss of mass does this represent in terms of the number of solar masses per second? (The mass of the Sun,1 $\mathrm{M}_\odot$, is about 2×10^{30} kg .)

(d) If the Sun expires as a red-giant when 10% of the mass of the Sun is converted into helium (the Schonberg-Chandrasekhar limit), work out the lifetime of the Sun.

{ (a) 26.3 MeV; (b) $6.2 \times 10^{11}\mathrm{kg\ s^{-1}}$; (c) $3.1\times 10^{-19}\mathrm{M}_\odot\mathrm{s}^{-1}$; (d) 10^{10} years }

A14. The supernova explosion SN1987A was observed on 23rd January 1987 in the Large Magellanic Cloud (LMC), a nearby galaxy only 160,000 light years from the Solar System, and was the brightest and best studied supernova since 1604. A pulse of 20 neutrinos from SN1987A was detected by 2 experiments. All 20 neutrinos arrived within a time interval of 10 s and their energies ranged from 7.5 to 40 MeV.

(a) If the neutrino has mass m, derive an *exact* expression for the time it takes a neutrino of total energy E to travel from the LMC to a detector on Earth. Derive an *approximate* expression for the time, in the case where the total energy of the neutrino is very much larger than its rest mass energy.

(b) From laboratory experiments based on studies of tritium-decay, the neutrino mass is known to be very small: < 7 eV/c^2 . Use the supernova data to obtain an alternative approximate *upper* limit to the mass of the neutrino by assuming that the high energy neutrinos (40 MeV) arrive first and the low energy neutrinos (7.5 MeV) arrive last.

{(a)$t = \dfrac{d}{c\sqrt{\left(1-\frac{m^2c^4}{E^2}\right)}}, t \approx \dfrac{d}{c\left(1-\frac{m^2c^4}{2E^2}\right)}$; (b) 15.2 eV/ c^2}

Several further questions on special relativity from past Tripos papers (not compulsory):

2009 B8: {Answer: the star is 4 light-years away; clock on Ship B reads 18 years when it gets the message; ship B arrives home when Earth time is 45 years; clock on ship A is ahead of that on ship B by 5 years.}

2006 B9 (Answer: 25 days after start, 15 light-days from Earth; 64 days after start; 4 days' supply left.)

2003 B8: {Answer: $f/3$; $3t$}

2008 B8: {Answer: $3c/5$; $4c/5$; 1.5 GHz}

2007 B7: {Answer: $0.061r/c$; $0.71fMc/(0.75fM+m)$}

2005 B7: {Answer: $\frac{3m}{\sqrt{2}} = 2.12$ m ; $\left(\frac{3}{2}\right)mc^2, \left(\frac{3}{4}\right)mc^2; \left(\frac{3}{4}\right)mc, 71^\circ$ }

2000 C13: {Answer: $4c/5$, $\sqrt{7}c/4$}

Please email me (slw55@cam.ac.uk) with any corrections or suggestions.