Waves and Quantum Waves

Question Sheet

1. Two points on a string at x_1 = 0 and x_2 = 1 m are observed as a travelling harmonic wave passes. The transverse displacements of the two points are:

   $y_1 = 0.2\sin(3\pi t)$ and $y_2 = 0.2\sin(3\pi t + \pi/8)$

   Deduce the wavelength, speed and direction of travel of the wave. Are your conclusions unambiguous?

2. A string has mass per unit length ρ = 0.1 kg m $^{-1}$ and is placed under a tension T = 10 N. The particle displacement of a wave at position x and time t is given by $A\cos(\omega t - kx)$, where A = 0.05m and ω = 20 π s^{-1}.

(a) Deduce the value of wavevector k and sketch (on a single, clearly labelled diagram) the particle displacement as a function of x at times t = 0 and t = 0.01 s.

(b) Sketch (on a single, clearly labelled diagram) the particle displacement as a function of time for position x = 0 and x = 0.1 m.

(c) Write down an expression for the particle transverse velocity as a function of x and t.

(d) Sketch (on a single, clearly labelled diagram) the particle transverse velocity as a function of x at times t = 0 and t = 0.01 s.

(e) Find expressions for the kinetic energy density and the potential energy density as functions of x and t and show that they are equal. Deduce the time average of the total energy density and hence find the average power transmitted along the string.

3. A steel piano wire is 0.7 m long and has a mass of 5 g. It is stretched with a tension of 500 N. What is the speed of transverse waves on the wire? To reduce the wave speed by a factor of 2 without changing the tension, what mass of copper wire would have to be wrapped around the wire?

4. A stretched membrane is lying in the xy-plane. The displacement, ψ, at time, t, produced by a wave propagating across the surface at an angle θ to the x-axis has the form

$$\psi(x, y, t) = A\cos(\omega t - k_x x - k_y y + \phi)$$

The wave speed is 10.0 ms^{-1}, k_x = 3.0 m^{-1}, k_y = 2.0 m^{-1} and the displacement at the origin is given by $\psi(0,0,t) = 0.05\sin\omega t$ [m].

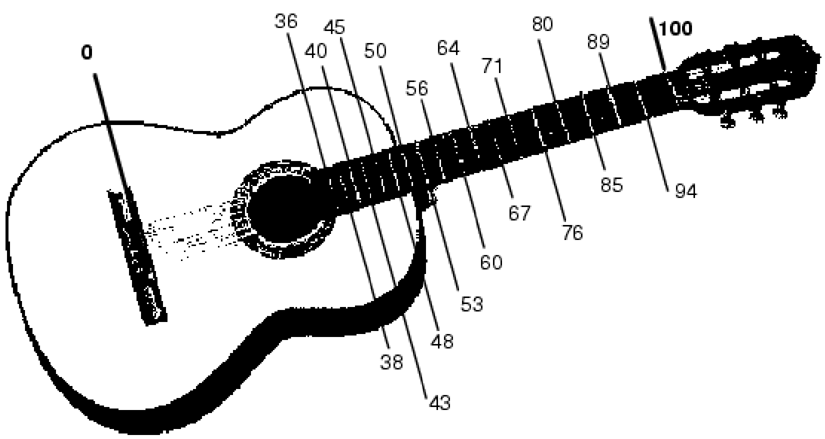
(a) Find the angle θ, the wavelength, the angular frequency ω, the amplitude A and phase ϕ.

(b) For what values of x will the displacement, $\psi(x, 0, t)$, be the same as that at the origin?

(c) Make a labelled sketch of the positions in the xy-plane of the lines of zero displacement at t=0.

5. An elastic string has mass 1 g and natural length 0.1 m. A mass of 1 kg is attached to its lower end, and the string is stretched by 0.02 m. Calculate the speed of propagation of transverse waves along the string.
   If the mass is set into small vertical oscillations, it oscillates with period T. A transverse ripple is now excited and travels along the string, and is reflected at both ends. How many times will this ripple pass a point on the string in the time T?
   (You may assume the tension in the string is uniform.

6. Guitarists usually press strings firmly down over a fret with their fingers. However, it is possible to obtain different notes by just touching a string above certain frets (and serious violinists exploit this). One octave is a factor of two in pitch (or frequency); there are 12 notes in an octave and each note is separated from the one above it by the same factor in pitch. Explain why touching the string above the fret at $\sim$67% gives a note one octave higher than that obtained when holding the string firmly against this fret. What are the corresponding ratios of pitch expected for the two fingerings for each of the frets at: near 80%, near 76%, and near 40%?



7. The transverse displacement $\psi(x, y, t)$ of a vibrating membrane satisfies the 2-D wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2},$$

where c is the wave speed.

(a) Show by direct substitution that the displacement

$$\psi(x, y, t) = A \sin(k_x x) \sin(k_y y) \cos(\omega t)$$

satisfies the 2-D wave equation and hence derive the connection between ω, k_x, k_y and c.

(b) A drumskin is rigidly held on the edges of a square of side a. Deduce the allowed values of k_x, k_y, and corresponding frequencies $f = \omega/2\pi$, for transverse vibrations of the drumskin.

(c) List the four lowest frequencies of vibration of the drumskin and, for each frequency, sketch all the corresponding vibration patterns. (Indicate the positions of nodes and anti-nodes, and the relative sign of the displacement over the area of the drumskin).

8. Ocean waves can travel in any direction, but waves breaking on the sea-shore are usually approximately parallel to the line of the beach. Use Huygen's Principle to explain this phenomenon, given that the velocity of the ocean waves decreases as the water depth decreases.

9. (a) A biconvex lens has faces with equal radii of curvature of 0.50 m. When an object is placed on the optic axis 0.75 m from the lens, a real image is formed which is exactly twice the size of the object. Calculate the refractive index of the glass.

(b) A thin lens of refractive index 1.5 has one convex side with a radius of magnitude 20 cm. When an object 1.00 cm in height is placed 50 cm from this lens, an upright image 2.15 cm in height can be seen. (i) Calculate the radius of the second side of the lens. Is it concave or convex?
(ii) Draw a sketch of the lens.

10. (a) Two lenses are separated by 35 cm. An object is 20 cm to the left of the first lens.
(i) The two lenses are converging lenses, each of focal length 10 cm. Find the position of the final image using both a ray diagram and the thin-lens equation. Is the image real or virtual? Upright or inverted? What is the overall lateral magnification of the image?
(ii) Now the first lens is a converging lens of focal length 10 cm and the second is a diverging lens of focal length 15 cm. Repeat the analysis carried out in part (i).

(b) Show that the minimum distance between an object and its real image formed by a simple convex lens is four times the focal length of the lens.

11. Two thin lenses of focal lengths $\mathrm{f_1}$ and $\mathrm{f_2}$ are placed very close to each other. Show that the focal length of the combination is given by

$$\frac{1}{\mathrm{f}}=\frac{1}{\mathrm{f_1}}+\frac{1}{\mathrm{f_2}}$$

(You might consider the image of the first lens as the object for the second lens)

An achromatic lens (that is, one which has the same focal length for different colours) is to be made by combining plano-convex and plano-concave lenses made from two different types of glass, A and B. These have the following refractive indexes:

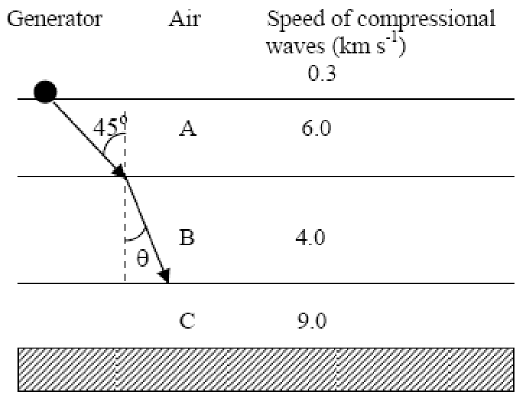
	A	B
red light	1.51	1.64
blue light	1.53	1.68

What radii of curvature should be used to produce a combination equivalent to a converging lens with a focal length of 500 mm?

12. A system of 8 slits, each separated from its neighbour by 0.05 mm, is illuminated with light of wavelength 576 nm. Using phasor analysis, evaluate at what angle from the normal there is the first minimum in intensity arriving on a distant screen. Find also the angles for the next two minima in intensity.

13. A ship's transmitter (operating at radio wavelengths greater than 3 m) loses contact with a land based receiving station, of height 150 m above sea level, at a distance of 2 km due to destructive interference between the directed and reflected waves. If the transmitter on the ship's mast is 25 m above the sea, what wavelength is the signal?
(Hint: there is a phase change of π of the wave upon reflection at the sea-water interface.)

14. The diagram shows the setup for an oil exploration test. A beam of longitudinal waves, generated at the surface, is directed through various layers of the earth's crust. The frequency of the waves is 75 Hz.



Calculate the wavelength of the waves in layer A and the angle θ to the vertical at which the waves enter layer B. Explain why waves do not penetrate layer C.

15. Monochromatic light of wavelength λ is incident at an angle θ to the normal onto a diffraction grating with 1200 lines/mm. Two successive maxima are seen in the diffraction pattern at angles of 14° and 73° to the normal to the diffraction grating. Calculate λ. In what other direction(s) should diffracted beams be seen?

16. Calculate the angular separation of the mercury yellow lines (577.0 and 579.1 nm) in the second order spectrum formed by a plane transmission diffraction grating having 1000 lines/cm, with light arriving at normal incidence. How wide should the light beam illuminating the grating be to resolve these two yellow mercury lines?

17. (a) Two narrow slits are separated by a distance d. Their interference pattern is to be observed on a screen a large distance L away. Light is incident normal to the vertical plane containing the two slits. Derive an expression for the positions of the maxima on the screen for light of wavelength λ.

Light of wavelength 650 nm from a laser is incident normal to the vertical plane containing two slits. The first interference maximum is 84 cm from the central maximum on a screen 12 m away. Find the separation of the slits. What is the total number of interference maxima that could, in principle, be observed? One of the slits is now covered by a thin plastic film which does not affect the amplitude of the light passing through it but changes the phase by π. Using phasor diagrams, deduce how this will change the diffraction pattern.

(b) A diffraction grating of 4800 slits per centimetre is illuminated with white light (wavelength range 400 to 700 nm). For how many orders can one observe the complete spectrum in the transmitted light? Do any of these orders overlap? If so, describe the overlapping regions.

18. (a) Light of wavelength 580 nm is incident on a single slit of width 0.30 mm. An observing screen is placed 2 m from the slit. Find the positions of the first dark fringes and the width of the central bright fringe.
(b) Using a phasor analysis discuss what happens if the middle third of the slit is now blocked.

19. The probability of finding a particle between position x and $x + dx$ is $P(x)dx = |\psi(x)|^2 dx$. Show that the mean x position of a particle trapped in an infinite potential well of width a in the second energy level is at the midpoint in x of the well. How does this compare to the most likely position of the particle?

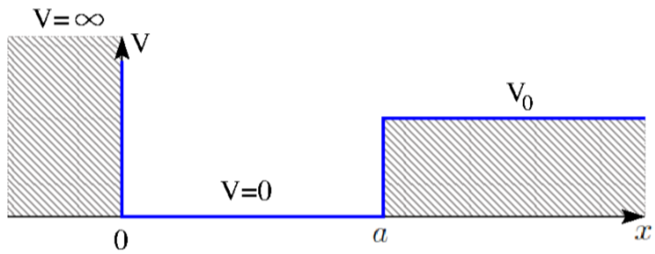
20. A particle in a one-dimensional box of length L is in the ground state. Find the probability of finding the particle in the region $0 < x < L/4$.

21. An electron is in a two-dimensional square box of length 0.1 nm. Find the ground state energy and the energies of the next 7 higher energy levels in electron volts. Find the wavelength of the photon emitted for transitions from each of these higher energy levels to the ground state.

22. A beam of electrons travelling in the $+x$ direction with energy E approaches a potential step of height V $(> \mathrm{E})$ located at x=0. If the electron amplitude just to the right of the step is C, evaluate the probability density functions $P(x)$ (defined at each position x as $P(x) = |\psi(x)|^2$) for the incident, reflected, and transmitted waves in terms of the relevant wave numbers and C.

Hence find an expression for the penetration depth x_{p} of the particle into the potential step, defined as the distance over which $P(x)$ falls to 1/e of its initial value.
(The variation with time t can be assumed to be implicitly suppressed in this question for simplicity, and you can ignore the interference between incident and reflected waves.)

23. A helium atom sits bound inside a potential of the form



Write the general form of the solution to the Schrödinger equation in each of the three regions:

(1) $0 < x < a$ (with wavevector k_1),

(2) $x > a$ (with wavevector $k_2 = i\kappa_2$), and

(3) $x < 0$,

defining the wavevectors in terms of $\mathrm{E}, \mathrm{V}_0, \mathrm{m}_{\mathrm{He}}$. You can assume $\mathrm{E} < \mathrm{V}_0$.

(a) By using the two conditions for matching the wavefunction at the boundaries, show that the condition for a bound state to exist is $\tan k_1 a = k_1 / i k_2$.

(b) Explain why bound states exist only if $\lambda_1 < 4a$ where $\lambda_1 = 2\pi / k_1$. (Harder. You may find a graphical method is helpful).

Answers and Hints

Note: These have been carefully checked so if you get the wrong answer, think carefully about what you might be missing before assuming any mistake.

1 16 m, 24 m s^{-1}, $-x$ direction.

2 a) 2π e) 4.93W

3 265 m s^{-1}; 15 g

4 a) 33.7°, 1.74 m, 36 rad s^{-1}, 0.05 m, $-\pi/2$

b) 2.1 m

5 34.3 m s^{-1}; 81

6 4,3,2

7 $\frac{c}{2a}\left(\sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{10}\right)$

9 a) 1.5

b) –35 cm

10 a) i) 30 cm to right of 2nd lens, M=2; ii) 7.5 cm to left of second lens, M=0.5

11 95mm, 190 mm

12 1.44 × 10^{-3} rad

13 3.75 m

14 80 m, 28.1°

15 595 nm, –28.3 °

16 4.2 × 10^{-4} rad

17 a) 9.3 × 10^{-6} m; 29; c) 4 beams show spectral splitting (2 either side)

18 ± 3.86 mm from centre

20 0.091

21 74.6 eV, 186.8 eV, 186.8 eV, 298.9 eV, 373.7 eV, 373.7 eV, 484.9 eV, 484.9 eV; 11.1 nm, 5.5 nm, 4.1 nm, 3.0 nm

22 LHS: $P_{\text{refl}}(x) = \frac{\left(k_1^2+\kappa_2^2\right)}{4k_1^2} C^2 = P_{\text{inc}}(x)$, RHS: $P_{\text{trans}}(x) = C^2 \exp(-2\kappa_2 x)$ where C is the amplitude just to the right of the step, and $\kappa_2 = \sqrt{2m_e(V-E)}/\hbar$. Hence $x_p = 1/(2\kappa_2)$.

23 $x < 0,\ \psi = 0;\quad 0 < x < a,\ \psi = A\cos k_1 x + B\sin k_1 x\,;\quad x > a, \psi = C\exp -\kappa_2 x$

$k_1 = \sqrt{2m_{He}E}/\hbar, \quad \kappa_2 = \sqrt{2m_{He}(V_0 - E)}/\hbar,$

(b) plot tan $k_1 a$ vs $k_1 a$ and find where it crosses $-1/\kappa_2 a$ to get answer.

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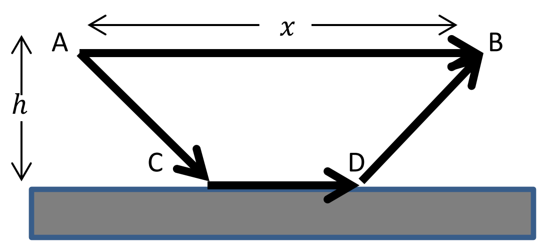
Additional (and some harder) Questions

1. A chain of length l is clamped at one end and hangs vertically under its own weight. The free end is given a sharp, sideways displacement and then returned to its original position. Describe what happens to the resulting wave pulse when it is reflected at the clamped end of the chain from its free end. How long does it take for the pulse to return to the bottom of the chain? (Hint: find the time by a suitable integration. The speed of sound for such a chain is given by $v = \sqrt{T/\rho}$ where T is the tension in the chain and ρ is the mass density/unit length.)

2. A and B are two points on the surface of the earth (which may be regarded as flat and parallel to the rock layer beneath).



An explosive charge is set off at A. The sound can reach B either directly along the surface soil, or indirectly via ACDB. In this latter case the sound suffers refractions at C and D and travels between C and D through the rock. If the speed of sound in the soil layer is v_1 and in the rock v_2 show that if the sound reaches B at the same time via the two paths, then

$$\frac{h}{x} = \frac{(1 - \sin\theta)}{2\cos\theta}$$

where $\sin\theta = v_1/v_2$ and h and x are as shown in the diagram.

3. A cubic box of volume Ω has rigid walls and contains a gas in which the velocity of sound c is independent of frequency. What is the lowest resonant frequency of the box? Show that the possible resonances have frequencies which are constant multiples of $(l^2 + m^2 + n^2)^{1/2}$, where l, m and n are integers.

Hence show that the number of different resonances with frequency less than ν is $\simeq \dfrac{4\pi\Omega\nu^3}{3c^3}$.

4. Explain, with the aid of ray diagrams, the following phenomena:

(a) You can use two plane mirrors to view an image of the back of your head.

(b) In hot weather dry streets can appear to be wet

(c) Secondary rainbows are sometimes seen, and the colour sequence in the secondary rainbow is opposite to that in the primary rainbow.

5. A diffraction grating of spacing d between slits is illuminated at normal incidence with light of wavelength λ. Using phasors, show that if a principal maximum occurs at an angle θ, then the angle between this maximum and the next minimum of intensity is $\Delta\theta = \lambda/(W\cos\theta)$ where $W = Nd$ is the total width of the grating.

For such a grating of finite width, comprising a large number N of slits, small subsidiary maxima are seen between the main maxima. Show that the ratio in the intensity between a main maximum and the next subsidiary one is approximately $9\pi^2/4$. You may assume that the first subsidiary maximum occurs when the phasors diagram wraps 1½ times.

Naturally occurring potassium is comprised mainly of ^{39}K (93%) and ^{41}K (6%) isotopes, for which the '*D1*' lines in their emission spectra have wavelengths of 770.10838 and 770.10792 nm respectively. Calculate the minimum width of the diffraction grating needed to separate the *D1* lines emitted by naturally occurring potassium and comment on the feasibility of making and using such a grating.

6. An extremely thin soap film forms a bubble which looks blue when viewed at normal incidence. Assuming that the film is so thin that constructive interference cannot occur for any thinner film, and that the refractive index of the soap film is 1.35, explain the origin of the colour and estimate the thickness of the film.

7. A parallel beam of light is incident normally on the longest face of a triangular prism with angles 174°, 3° and 3° and refractive index n= 1.6. What is the separation between neighbouring maxima on a far distant screen when light of wavelength 500nm is used? What is seen on the screen if white light is used? (You may assume that a beam of light passing through a small angle prism of angle α and refractive index n is deviated by an angle (n-1)α).

8. a) An electron beam is represented by the plane wave $\exp\{i(k_x x + k_y y)\}$. Calculate k_x and k_y if the electron energy is 10eV and the beam travels at 15° to the y-axis.
b) The beam is incident on a potential step perpendicular to the x-axis. The step is of height -5 eV (i.e. so as to slow the electrons). Show that the beam is totally reflected. Estimate the penetration of electrons into the potential step.

9. A particle is bound in a one-dimensional potential well. The potential is positive and very large for $x < 0$, attractive and equal to -V for $0 < x < a$, and zero for $x > a$. The total energy of the lowest state in the well is $-1/4V$. Show that the probability that the particle is outside the attractive part of the well is

$$\frac{9\sqrt{3}}{8\pi + 12\sqrt{3}}$$

Solutions

1 $4\sqrt{\frac{l}{g}}$

6 74nm

7 7.95×10^{-6} m

8 a) $k_x = 0.42 \times 10^{10}\,\mathrm{m}^{-1}$ and $k_y = 1.56 \times 10^{10}\,\mathrm{m}^{-1}$

b) 4.7×10^{-11} m